PARAMETER PLANE METHODS FOR LINEAR SYSTEMS WITH TIME DELAY

Anthony Ch. Hadjiconstandis

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

PARAMETER PLANE METHODS FOR
LINEAR SYSTEMS WITH TIME DELAY

by

Anthony .Ch. Hadjiconstandis

Thesis Advisor: G. J. Thaler

March 1973

Parameter Plane Methods
for
Linear Systems with Time Delay

by

Anthony .Ch. Hadjiconstandis
Lieutenant Commander, Hellenic Navy
B.S.E.E., U.S. Naval Postgraduate School, 1972

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
March 1973

ABSTRACT

After a brief introduction on the parameter plane methods, the behavior of simple systems with time delay has been initially studied by the root-loci method and then the state of the art of parameter plane methods with time delay has been presented.

A variation of the solution in the state of the art presentation has been derived and an appropriate digital computer program has been constructed. As an extension of studies in the parameter plane, a three variable parameter system has been approached by using basic descriptive geometry properties and simple illustrative examples have been worked out in the parameter space.

Finally, a real life system with time delay has been analysed and redesigned by parameter plane methods.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

ACKNOWLEDGEMENTS

The author is indebted to Dr. G. J. Thaler for the orientation he provided, and the helpful suggestions and encouragement.

The author is also indebted to A. Gerba for the helpful amendments suggested and J. Ward for his help and providing the initial attraction to the control field.

ACKNOWLEDGEMENTS

The author is indebted to Dr. G. J. Thaler for the orientation he provided, and the helpful suggestions and encouragement.

The author is also indebted to A. Gerba for the helpful amendments suggested and J. Ward for his help and providing the initial attraction to the control field.

DEDICATION

To my wife Stavroula for her patience, help and encouragement.

I. BASIC PARAMETER PLANE TECHNIQUES

A. GENERAL

Parameter plane techniques are algebraic methods designed to study the behavior of linear systems through their characteristic equation, when two of their parameters are adjustable. So in fact they are an extension of the Root-Locus technique where only one of the parameters of the system is adjusted to meet any desired performance requirements.

B. SILJAK'S APPROACH

Parameter plane techniques have been at first studied by Mitrovic who considered that the parameter variations directly affected two of the coefficients of the system's characteristic equation.

Then Siljak extended Mitrovic's work by including variations on the system's parameters "α" and "β" which appeared linearly in one or more of the coefficients of the characteristic equation, which was of the form:

$$F(s) = \sum_{k=0}^{n} a_k s^k = 0 \tag{I-1}$$

$$\text{where } a_k = b_k\alpha + c_k\beta + d_k \tag{I-1a}$$

with b_i, c_i, d_i constant and real

and α,β variable and real.

Curves obtained in the α,β plane are called parameter plane curves and they provide information about the roots of the system's characteristic equation in a similar way to root loci.

Siljak's work can be summarized in its goals as follows:

First make an appropriate substitution for s in equation (I-1). This substitution must be in terms of ζ and ω_n, or σ and ω, or in general in terms of quantities one wants to study.

Second step is to separate the result of equation (I-1) after the substitution is made, into real and imaginary parts thereby obtaining two equations in α, β, ζ, ω and the constants of the system.

Finally, one solves for α, β in terms of ζ and ω. If, for example, one fixes $\zeta = \zeta_o$ and varies ω over the range $0 < \omega < \infty$, one will obtain in the α-β plane the necessary relation between α and β in order to have a complex root pair of damping ratio $\zeta = \zeta_o$ (a constant ζ curve). If, on the other hand, one fixes $\omega = \omega_o$ and varies ζ over the range $-1 < \zeta < 1$, one will obtain a constant omega curve denoting the necessary α-β relation to provide a complex root pair of natural frequency $\omega = \omega_o$. Siljak used the following substitution in order to study constant ζ and constant ω curves:

$$s^k = \omega^k \left[T_k(-\zeta) + j\sqrt{1-\zeta^2}\, U_k(-\zeta) \right] \tag{I-2}$$

where U_k and T_k are Chebyshev functions of the first and second kind respectively, given by the following recursive relations and initial conditions:

$$\begin{aligned} T_{k+1} - 2\zeta T_k(\zeta) + T_{k-1}(\zeta) &= 0 \\ U_{k+1} - 2\zeta U_k(\zeta) + U_{k-1}(\zeta) &= 0 \end{aligned} \tag{I-3}$$

where

$$\begin{aligned} T_o(\zeta) &= 1 \\ T_1(\zeta) &= Z \end{aligned} \quad \text{and} \quad \begin{aligned} U_o(\zeta) &= 0 \\ U_1(\zeta) &= 1 \end{aligned} \tag{I-4}$$

It is to be noted that:

$$T_k(-\zeta) = (-1)^k \, T_k(\zeta)$$
$$U_k(-\zeta) = (-1)^{k+1} \, U_k(\zeta) \qquad \text{(I-5)}$$

In the second step after substituting equation (I-2), [considering also the results of equations (I-3) to (I-5)], into equation (I-1), Siljak obtained:

$$\sum_{k=0}^{n} (-1)^k \, a_k \omega^k \, U_{k-1}(\zeta) = 0 \quad \text{and}$$
$$\sum_{k=0}^{n} (-1)^k \, a_k \omega^k \, U_k \, (\zeta) = 0 \qquad \text{(I-6)}$$

$$\text{where } a_k = b_k\alpha + c_k\beta + d_k \qquad \text{(I-1a)}$$

So finally:

$$B_1\alpha + C_1\beta = -D_1$$
$$B_2\alpha + C_2\beta = -D_2 \qquad \text{(I-7)}$$

from which:

$$\alpha = \frac{C_1 \, D_2 - C_2 \, D_1}{B_1 \, C_2 - B_2 \, C_1} \qquad \beta = \frac{B_2 \, D_1 - B_1 \, D_2}{B_1 \, C_2 - B_2 \, C_1} \qquad \text{(I-8)}$$

where:

$$B_1 = \sum_{k=0}^{n} (-1)^k b_k \omega^k U_{k-1} \qquad B_2 = \sum_{k=0}^{n} (-1)^k b_k \omega^k U_k$$
$$C_1 = \sum_{k=0}^{n} (-1)^k c_k \omega^k U_{k-1} \qquad C_2 = \sum_{k=0}^{n} (-1)^k c_k \omega^k U_k$$
$$D_1 = \sum_{k=0}^{n} (-1)^k d_k \omega^k U_{k-1} \qquad D_2 = \sum_{k=0}^{n} (-1)^k d_k \omega^k U_k$$

The study for conditions in the α-β plane giving a real root at $s = \sigma$, was much easier. In fact, by substituting $s = \sigma$ in equation (I-1),

Siljak obtained

$$\sum_{k=0}^{n} a_k \sigma^k = 0 \qquad \text{(I-9)}$$

which by using (I-1a), led to:

$$\alpha \sum_{k=0}^{n} b_k \sigma^k + \beta \sum_{k=0}^{n} c_k \sigma^k + \sum_{k=0}^{n} d_k \sigma^k = 0$$

which is the equation of a straight line in the α-β plane, for given "σ". Details on this work can be found in reference [1].

C. HOLLISTER'S WORK

In section 4 of reference [2] Hollister obtained parameter plane equations for a system with characteristic equation of the form of equation (I-9), with:

$$\alpha_k = b_k A + c_k B + d_k AB + f_k \qquad \text{(I-10)}$$

His solution for constant "ω_n" and "Z" curves was obtained by simultaneously solving two equations of the form:

$$\begin{aligned} B_1 A + C_1 B + D_1 AB + F_1 &= 0 \\ B_2 A + C_2 B + D_2 AB + F_2 &= 0 \end{aligned} \qquad \text{(I-11)}$$

where B_i, C_i, D_i, F_i, with $i = [1,2]$, are functions of "ω_n" and "ζ", as well as of the first and second kind Tchebychef functions $T_k(\zeta)$ and $U_k(\zeta)$.

D. PRACTICAL CONSIDERATIONS (AN INDICATION)

Although the concepts governing analysis and synthesis of feedback control systems, once established are quite straightforward, there is some difficulty in making an appropriate selection of the range of A and B parameter values to be used in the digital computer program.

Some practical ways for defining a set of approximate ranges of the parameter values for use in the first run of a parameter plane computer program, are presented in section V.

II. SYSTEMS WITH TIME DELAY -- A RESUME

The purpose of this section is two-fold:

First: To introduce the reader to the State of the Art of analysis for systems with transport lag.

Second: To define the objectives of the work presented in this thesis.

Since stability analysis will be treated separately in a later section, items concerning this subject are not included here.

A. THE TRANSPORT LAG - AN INTRODUCTION

1. The existence of a time delay effect associated with any system, linear or non-linear, is inevitable since the working signal (command or information), needs a certain period of time to travel through and reach the output. So if a signal r(t) is applied in the input of one of the most elementary systems, as shown in figure 2-1, the output c(t) will be in general equal to:

$$c(t) = Kr(t - \tau) \qquad \text{(II-1)}$$

where τ is a time interval associated with the equivalent length of the system "ℓ" and the average speed $\bar{u}$ of the signal through the device. Specifically:

$$\tau = \frac{\ell}{\bar{u}} \qquad \text{(II-1a)}$$

There are of course cases when the dimensions of the system and the speed of the signal are such to imply a negligible time delay effect as for example in the actual lumped A.C. circuitry where the wavelength of the signal is large compared with the dimensions of the circuit-system.

SIMPLE LINEAR GAIN SYSTEM IN T-DOMAIN

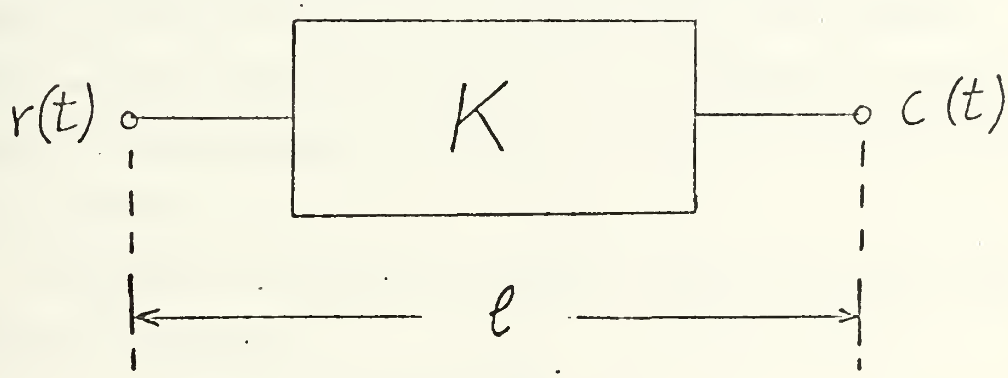


FIGURE 2-1

But there are also other cases when the time delay effect is important and should be taken into consideration.

In such a case one can define a system in the s-domain, equivalent to the one of figure 2-1 by Laplace-transforming equation (II-1):

$$C(s) = K\left(R(s)e^{-\tau s}\right) = \left(Ke^{-\tau s}\right)R(s)$$

$$\text{or} \quad \frac{C(s)}{R(s)} = Ke^{-\tau s} \qquad \text{(II-2)}$$

2. After this presentation one can define the time-delay effect as "the physical phenomenon of finite time spent by a signal travelling through a finite system." For each combination of signal and system there is assiciated a time delay constant "τ" which denotes the time necessary for this travel.

B. THE EFFECTS OF TRANSPORT LAG ON A SIMPLE LINEAR SYSTEM

Before the State of the Art in analyzing systems with transport lag in the parameter plane is discussed, it is interesting to observe the results of this time delay in a simple system by use of conventional methods of analysis.

1. Consider, for example, the system of figure 2-1a, with transfer functions as follows:

a. Open loop: $GH = \dfrac{K}{S(S+1)}$

b. Closed loop: $\dfrac{C}{R} = \dfrac{K}{S^2 + S + K}$

From the root locus analysis of this system as shown in figure 2-1b, a pair of roots at $Z = .707$ and $\omega_n = .707$ is derived for $K = \frac{1}{2}$.

2. In figure 2-2 the same system is shown with the exception that a time delay is now present in the forward path and the transfer function changes to:

SIMPLE DELAY-FREE SYSTEM AND ITS R.LOCUS

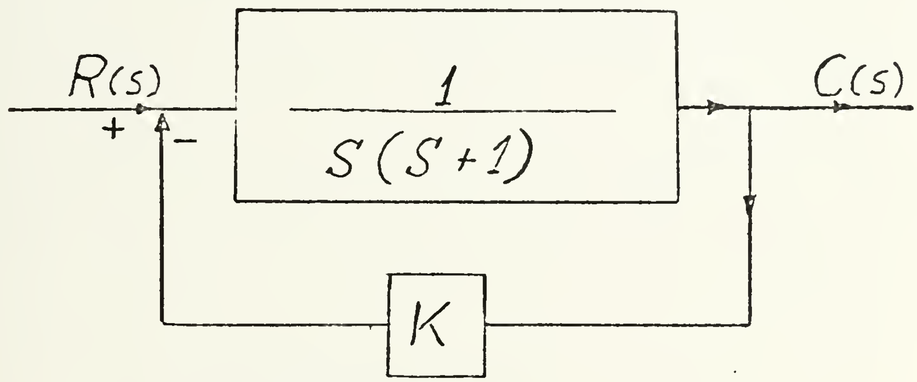


FIGURE 2-1a

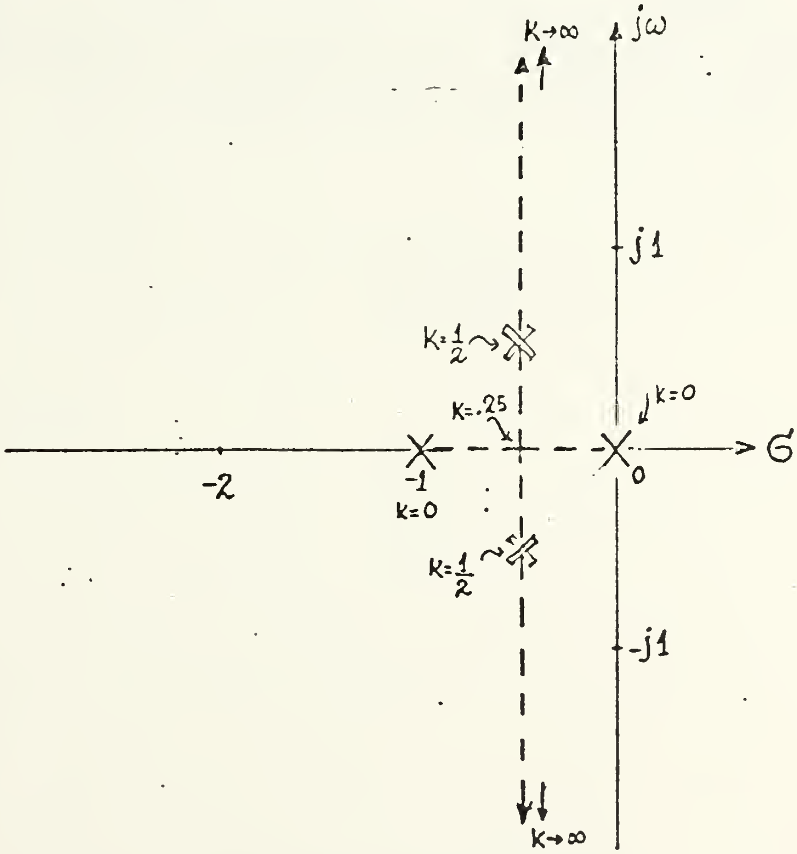


FIGURE 2-1b

BLOCK DIAGRAM FOR T-DELAY INCLUDED

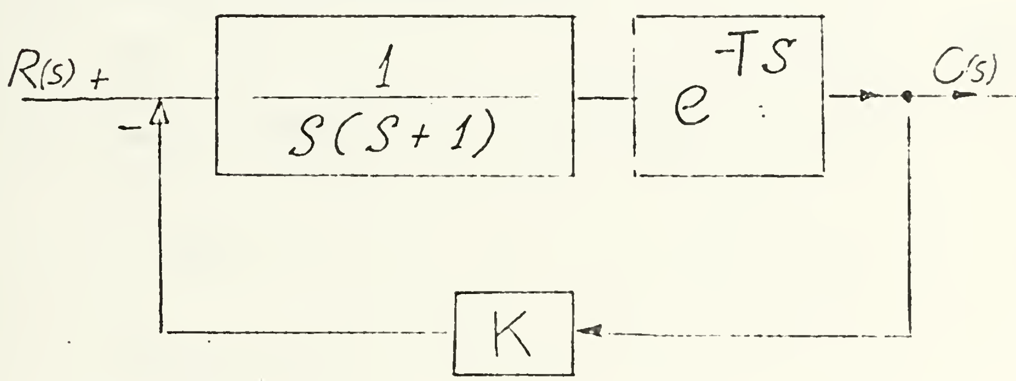


FIGURE 2-2

a. Open loop: $GH = \frac{K}{S(S+1)} e^{-\tau s}$ (II-3a)

b. Closed loop: $\frac{C}{R} = \frac{K e^{-\tau s}}{S^2 + S + Ke^{-\tau s}}$ (II-3b)

In trying to apply the root-locus analysis technique one is faced with the first difficulty since,

$$GH = \frac{K}{S(S+1)e^{Ts}}$$

has an infinite number of roots, as the series expansion of e^{Ts} implies.

3. There are a number of approximation techniques one can apply to overcome this difficulty as mentioned below:

a. For small T and low frequency,

$$e^{sT} = 1 + sT \qquad \text{(II-4)}$$

b. From the fact that

$$e = \lim_{v \to \infty} \left(1 + \frac{1}{v}\right)^v$$

and

$$e^{-\alpha} = \lim_{v \to \infty} \left(\frac{1}{1 + \frac{\alpha}{v}}\right)^v$$

one can approximate:

$$e^{-sT} = \left(\frac{1}{1 + \frac{sT}{v}}\right)^v$$ if v is sufficiently large. In order to have an intuitive feeling of how large v should be for acceptable results, note that $(1 + 1/10)^{10} = 2.6$ producing a deviation of 4.4% from e = 2.72. So, any v < 4 will produce too crude an approximation since $(1 + \frac{1}{4})^4 = 2.44$, and 2.44 ÷ 2.72 = 9%.

c. In the case where both the frequency and T are too large to neglect in the expansion of e^{-Ts}, it is possible to approximate the

time delay term by the ratio of two polynomials in "s" yielding to a certain order of terms identical with those contained in the series. That is,

$$e^{-Ts} = F_{a,b}(T\cdot s) = \frac{N_{a,b}(+Ts)}{D_{a,b}(+Ts)}$$

where "a" and "b" are the order of the numerator and denominator polynomials. So since

$$e^{-Ts} = 1 - Ts + \frac{T^2}{2}\cdot s^2 - \frac{T^3}{6}\cdot s^3 + \frac{T^4}{24}\cdot s^4 - \ldots$$

if third order terms can be neglected, it can be shown by straightforward division and a series of comparisons with the series terms that a convenient approximation is the following:

$$e^{-Ts} = \frac{1 + \lambda_1 s + \lambda_2 s^2}{1 + \mu_1 s} = \frac{1 - 3/4(Ts) + T^2/4(s^2)}{1 + T/4(s)}$$

These are the basic features of the so-called "Padé Approximation Technique." In appendix A the most often used approximations are listed (degree of higher order polynomial equal to 12). In this introductory study of time delay effects the Padé approximation will be selected as the most accurate. From appendix A then,

$$e^{-(Ts)} = F_{4,4}(Ts) = \frac{N_{4,4}(Ts)}{D_{4,4}(Ts)} =$$

$$\frac{1680 - 840Ts + 180T^2s^2 - 20T^3s^3 + T^4s^4}{1680 + 840Ts + 180T^2s^2 + 20T^3s^3 + T^4s^4},$$

and by use of this approximation the closed loop transfer function of the system becomes:

$$\frac{C}{R} = \frac{K\cdot N_{4,4}(Ts)}{(s^2+s)\cdot D_{4,4}(Ts) + K\cdot N_{4,4}(Ts)} \qquad \text{(II-5)}$$

4. The root loci of this system have been plotted for four different values of T by use of a digital computer, as the following table indicates:

TABLE 2-1

T = 0	Figure 2-1b
T = .1	Figure 2-2a, 2-3a
T = .2	Figure 2-2b, 2-3b
T = .3	Figure 2-2c, 2-3c
T =1.0	Figure 2-2d, 2-3d

It should be noted that the Padé approximation introduces in general, roots in the system which all-together do not disturb seriously the dominant roots near the origin. Data on the characteristic equation resulting for each of the above "T's" are shown in Table A2 of Appendix A.

5. <u>Partial Conclusions</u>

a. A careful observation of figures 2-1b, 2-2, 2-3, and Table 2-1 shows some of the effects of the existence of transport lag in the linear system under consideration, namely:

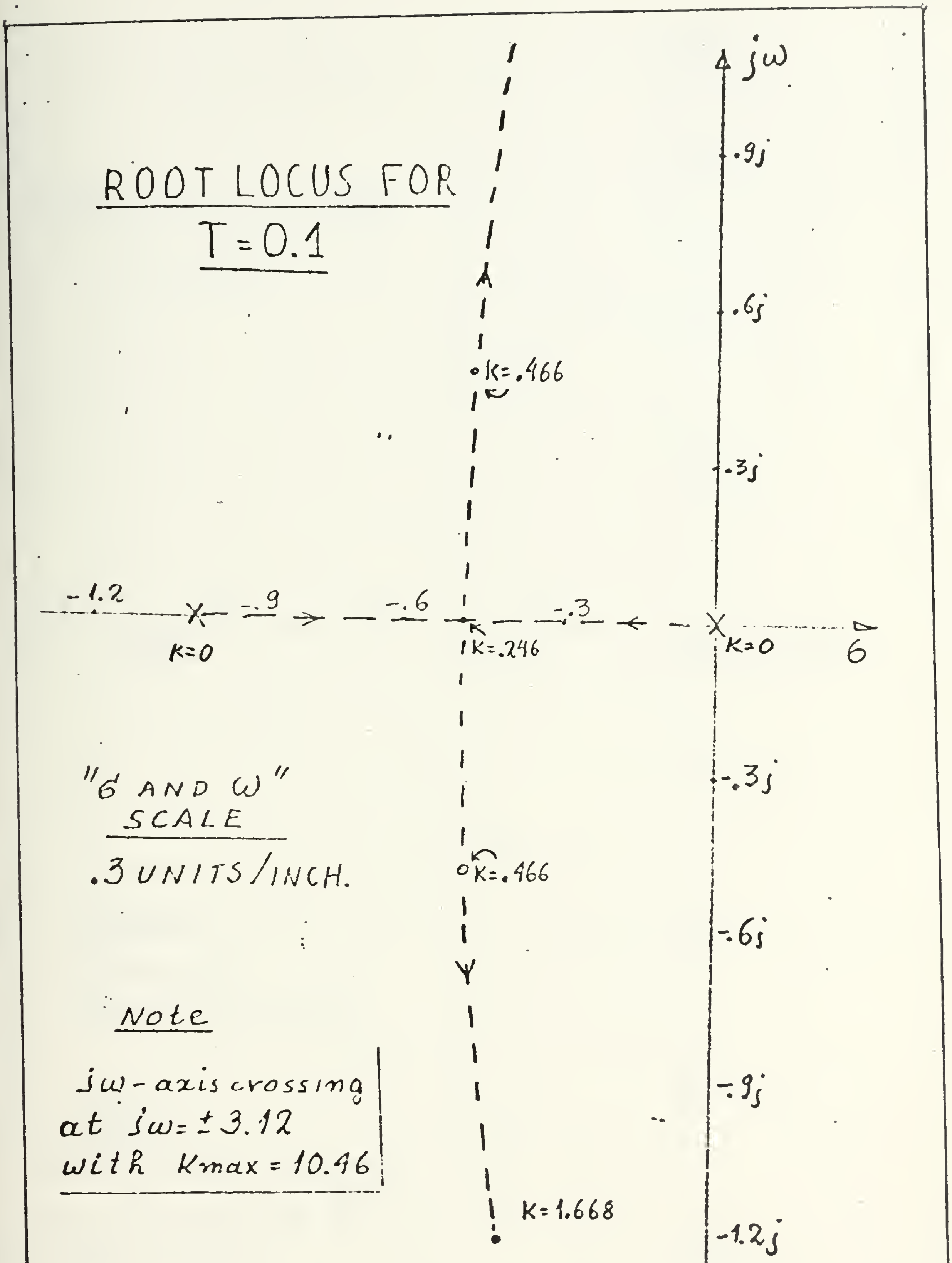
(1) With the insertion of time delay in the system the segment of the root locus parallel to the $j\omega$ axis doesn't exist, but with increasing time delay constant the corresponding segment is bent analogously toward the $j\omega$ axis. In other words the system becomes unstable.

(2) As a consequence the K_{max} of the system which maintains absolute stability is decreased with increasing time delay.

(3) It is also obvious (Table 2-1), that the relative stability is reduced with increasing time delay constant.

TABLE 2-2

T	$K(\zeta=0)$	$K(\zeta=.707)$	$K_{max}(\zeta=1.0)$	SYSTEM IN "ABS. STABILITY" YES	NO	"$j\omega$-axis" crossing	Phase Margin	Gain Margin
0.0	.250	.500	∞	X		∞	69°	∞
0.1	.246	.466	10.46		X	$\pm j3.12$	—	22.2
0.2	.233	.420	5.32		X	$\pm j2.18$	—	12.7
0.3	.225	.400	3.50		X	$\pm j1.74$	—	8.75
1.0	.164	.255	1.178		X	$\pm j0.86$	—	4.62



ROOT LOCUS FOR
T = 0.1
jω
.9j
.6j
K=.466
.3j
-1.2
-.9
-.6
-.3
K=0
K=.246
K=0
σ
-.3j
"σ AND ω"
SCALE
.3 UNITS/INCH.
K=.466
-.6j
Note
jω-axis crossing
at jω= ± 3.12
with Kmax = 10.46
-.9j
K=1.668
-1.2j

FIGURE 2-2a

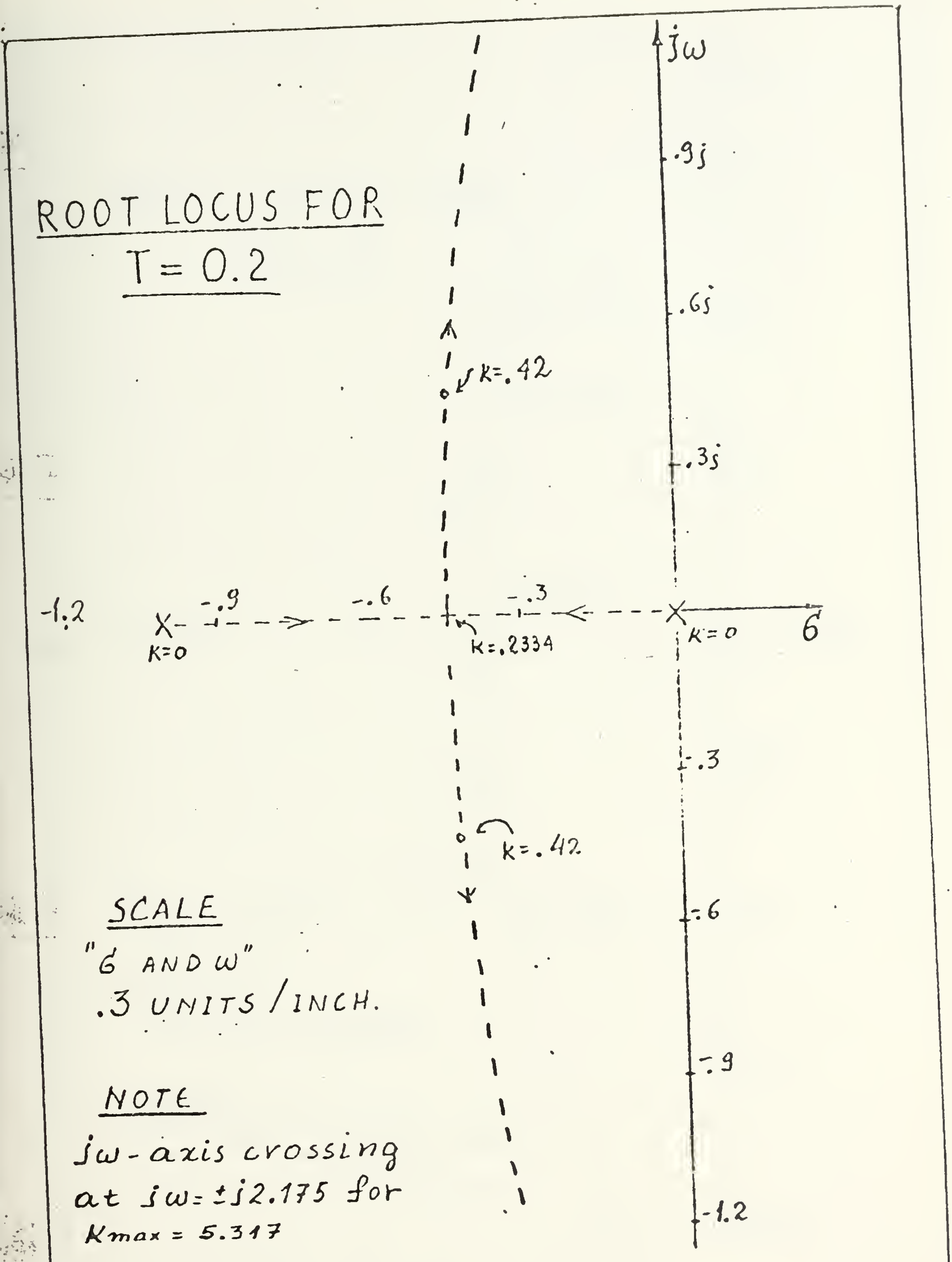

ROOT LOCUS FOR
T = 0.2
jω
.9j
.6j
K=.42
.3j
-1.2
-.9
-.6
-.3
K=0
K=.2334
K=0
σ
-.3
K=.42
-.6
-.9
-1.2
SCALE
"σ AND ω"
.3 UNITS / INCH.
NOTE
jω-axis crossing
at jω=±j2.175 for
Kmax = 5.317

FIGURE 2-2b

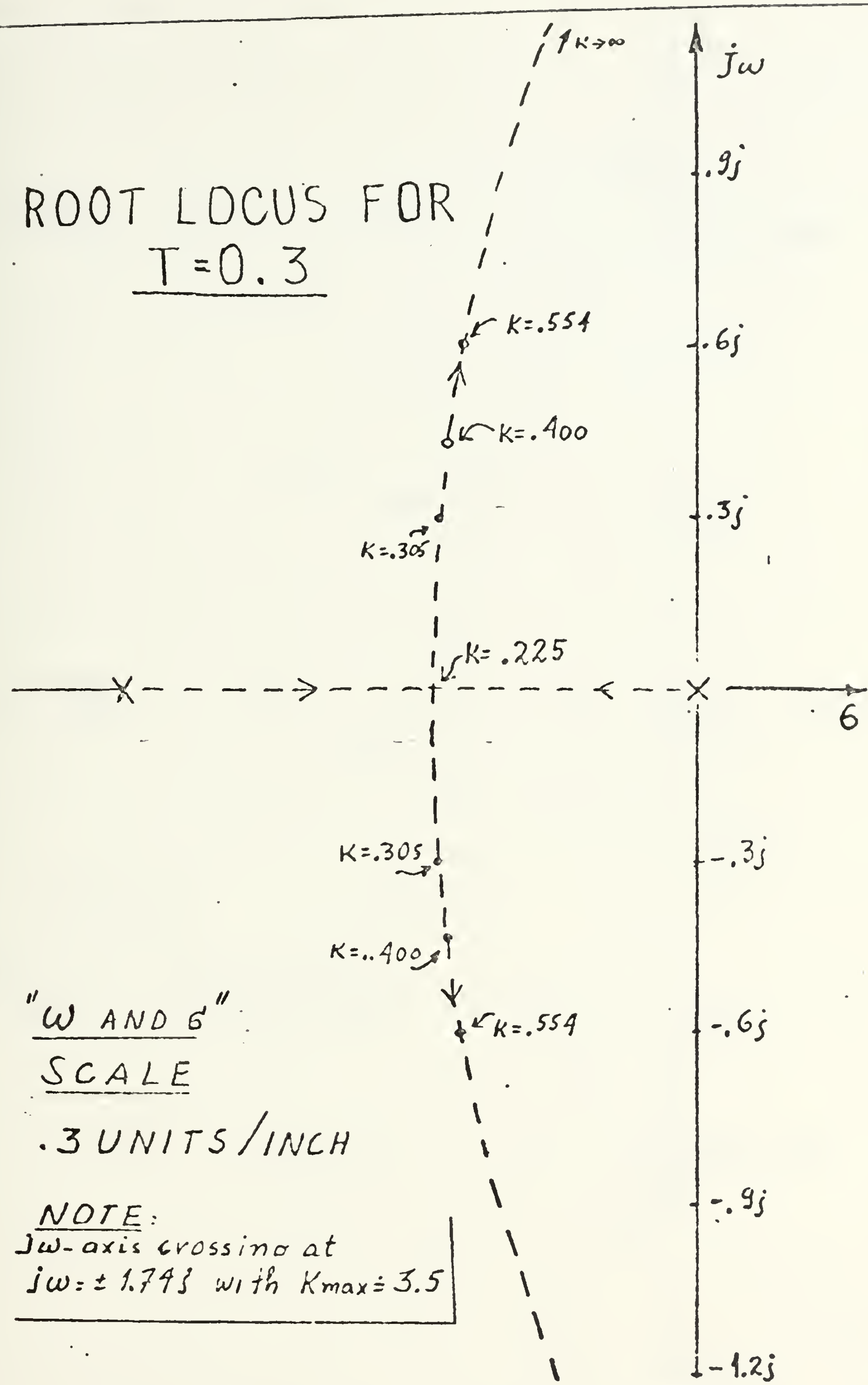
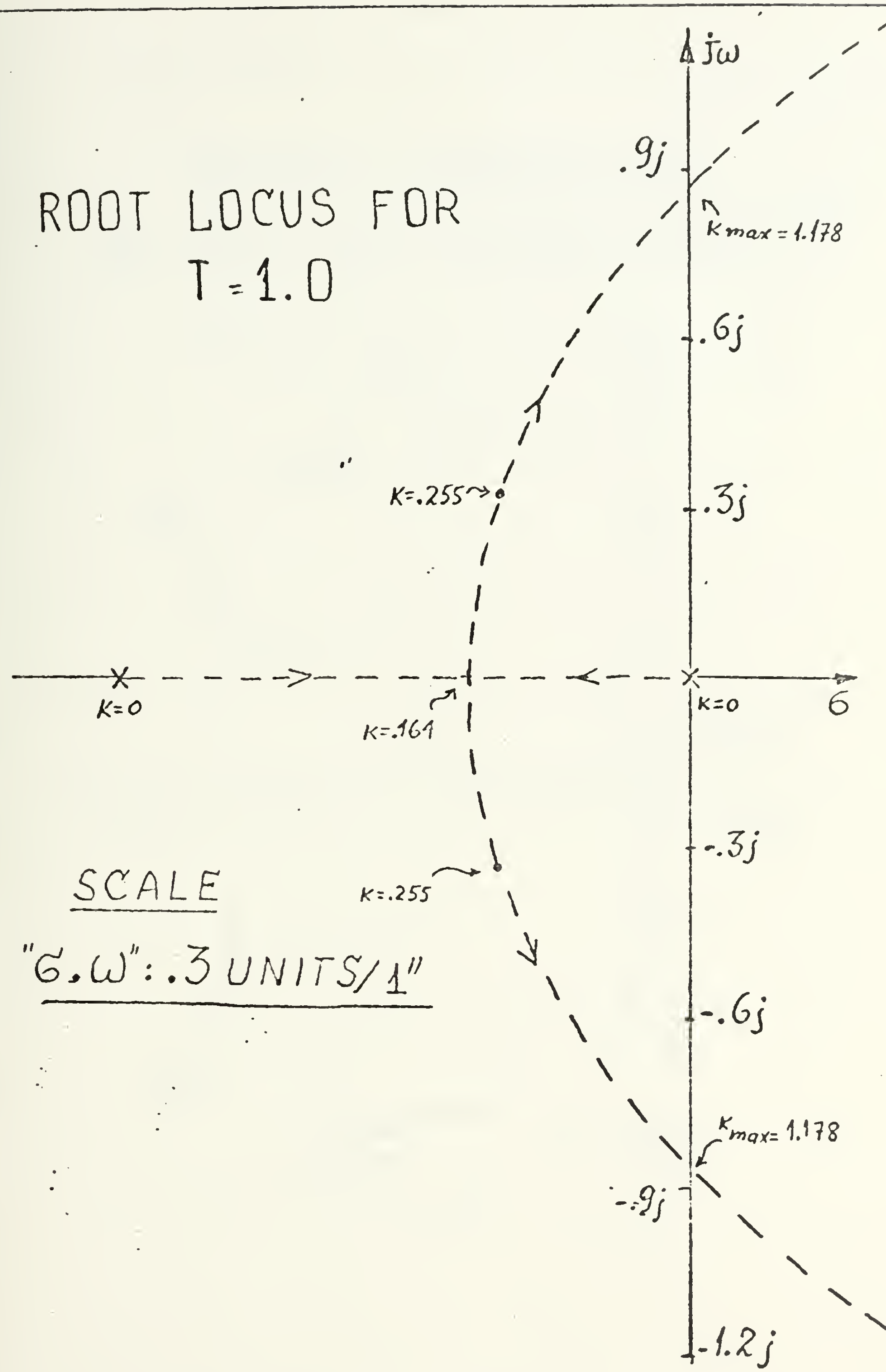

ROOT LOCUS FOR
T=0.3
K→∞
jω
.9j
K=.554
.6j
K=.400
.3j
K=.305
K=.225
6
K=.305
-.3j
K=..400
K=.554
-.6j
-.9j
-1.2j
"ω AND σ" SCALE
.3 UNITS/INCH
NOTE:
Jω-axis crossing at
jω=± 1.74j with Kmax=3.5

FIGURE 2-2c



ROOT LOCUS FOR
T = 1.0
jω
.9j
K max = 1.178
.6j
K=.255
.3j
K=0
K=.161
K=0
6
-.3j
SCALE
"6,ω": .3 UNITS/1"
K=.255
-.6j
K max= 1.178
-.9j
-1.2j

FIGURE 2-2d

ROOTS INTRODUCED BY THE PADÉ APPROXIMATION

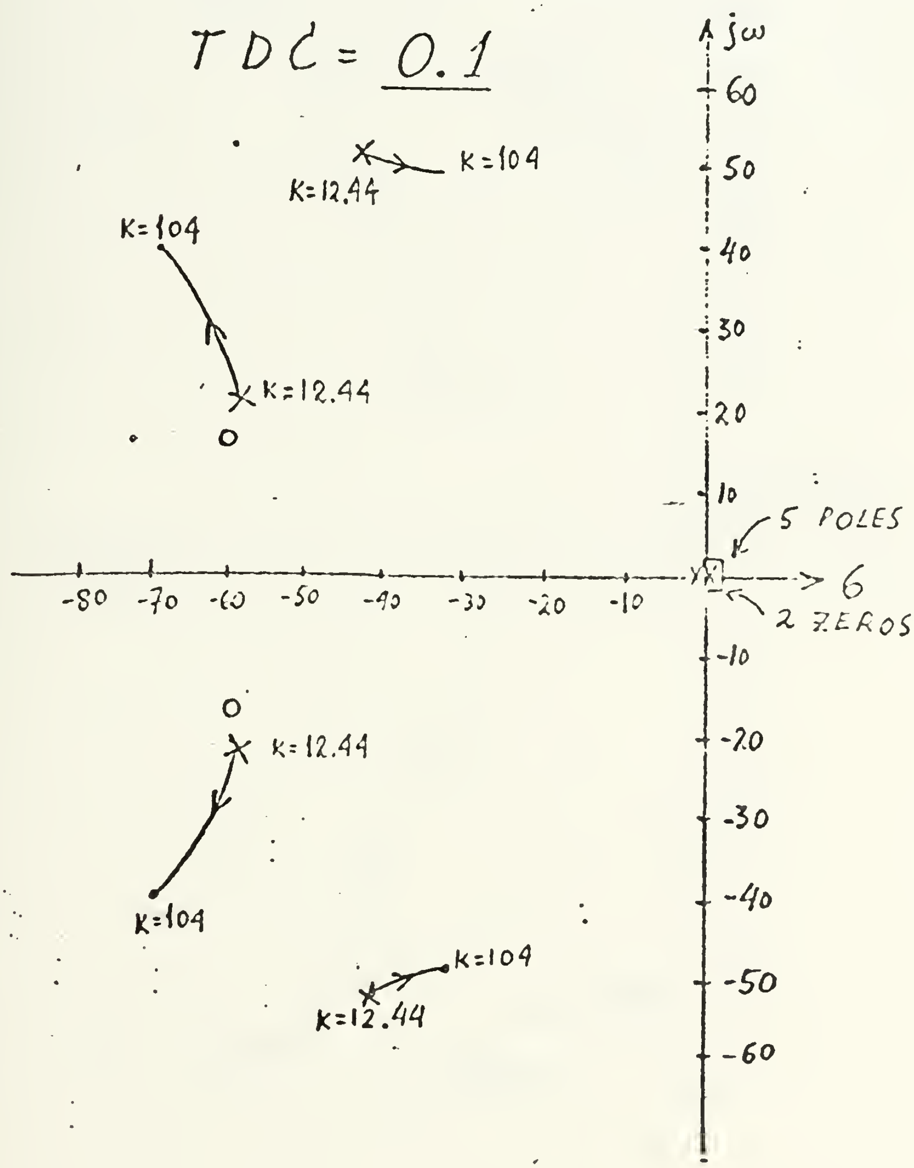


FIGURE 2-3a

ROOTS INTRODUCED BY THE PADÉ APPROXIMATION

TDC = 0.2

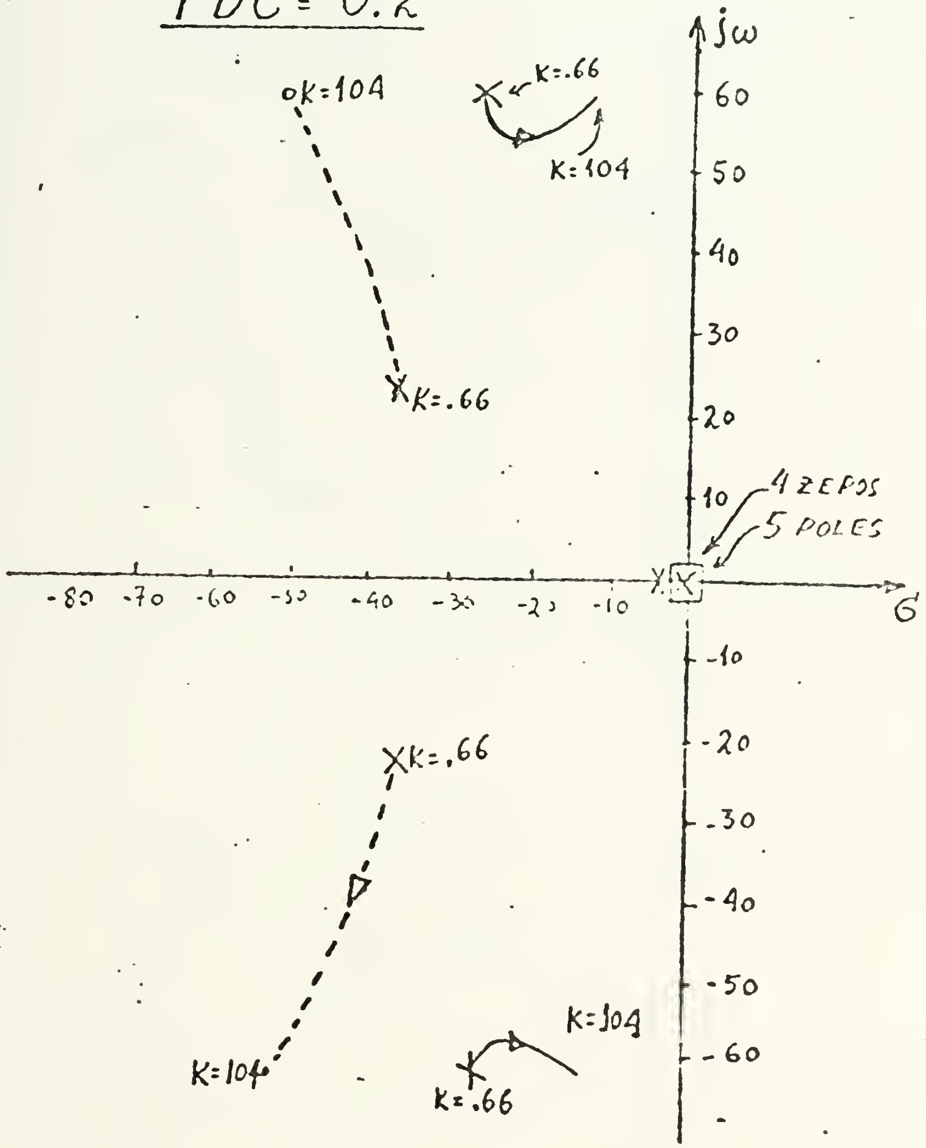


FIGURE 2-3b

ROOTS INTRODUCED

BY THE PADÉ APPROXIMATION

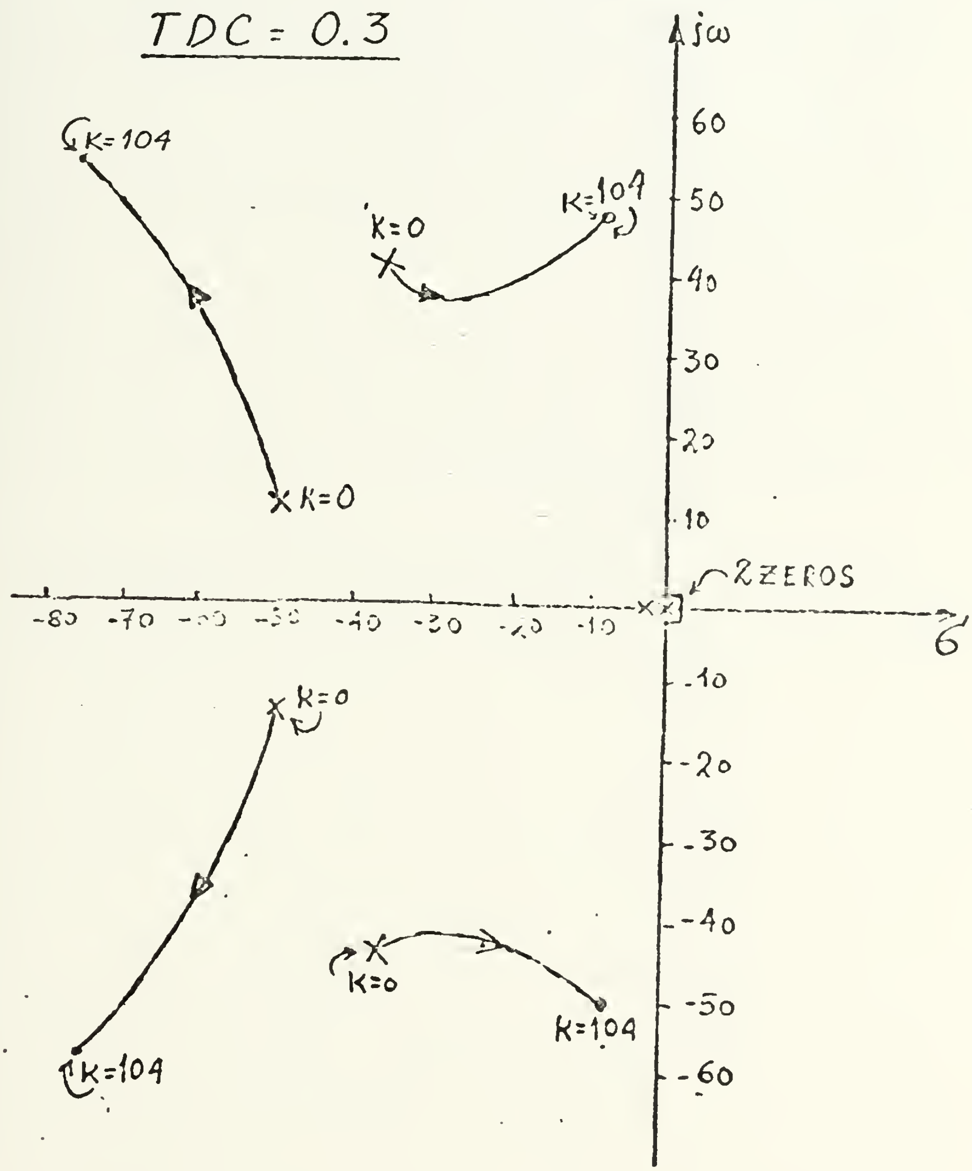


FIGURE 2-3c

ROOTS INTRODUCED BY THE PADÉ APPROXIMATION

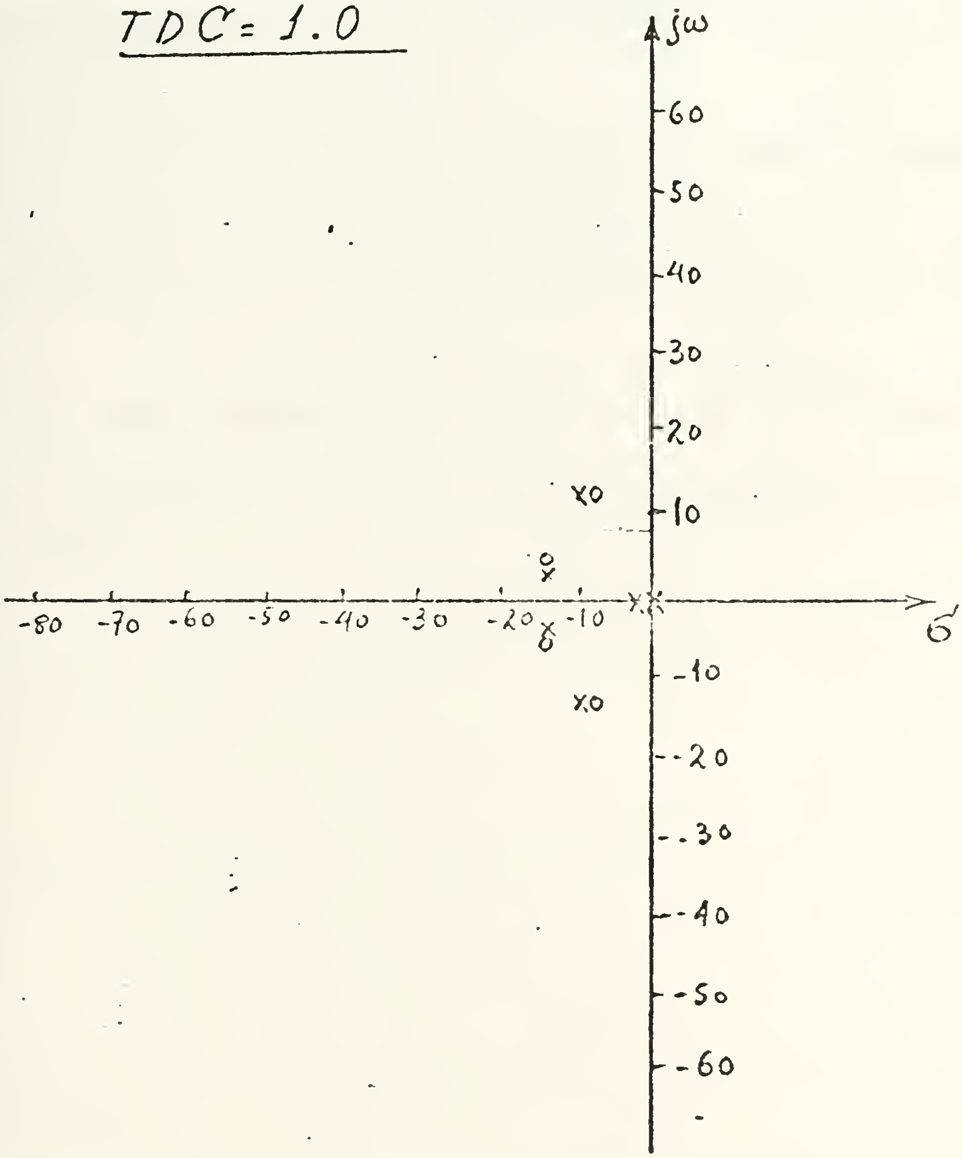


FIGURE 2-3d

6. General Conclusions

From this discussion, the main facts concerning the insertion of a time delay in a linear system are apparent:

a. Any system carries a time-delay effect with it. Therefore, no system can be thought of as absolutely stable. Nevertheless, systems with very small time delay deviate so little from the ideal case, that they can be thought of as delay-free.

b. By considering Figures 2-3 we see that when approximating the time delay term by the Padé technique, poles and zeroes are introduced at finite ω_n, in the "s" domain. This is true for any other approximation.

c. For T increasing, the relative stability of the system decreases and the root locus suffers an increasing deformation towards the $+\sigma$ direction.

C. PARAMETER PLANE METHODS FOR SYSTEMS WITH TIME DELAY (STATE OF THE ART)

1. The closed loop gain of the system shown in figure 2-4a is

$$\frac{C}{R}(s) = \frac{N(s)}{D(s)\epsilon^{Ts} + N(s)}$$

with

$$D(s) = \sum_{k=0}^{n} d_k s^k \quad \text{and} \quad N(s) = \sum_{k=0}^{J} n_k s^k$$

So the characteristic equation is

$$\epsilon^{Ts} \sum_{k=0}^{n} d_k s^k + \sum_{k=0}^{J} n_k s^k = \sum_{k=0}^{n} a_k(s) s^k = 0 \tag{II-6}$$

where $a_k(s)$ is a function of ϵ^{Ts}, and contains the variable parameters A,B of the system. So if only linear combinations of a and b are considered:

$$a_k(s) = Ab_k + ac_k\epsilon^{Ts} + Bd_k + be_k\epsilon^{Ts} + f_k + g_k\epsilon^{Ts} \tag{II-7}$$

2. Intending to derive constant "ζ" and "ω_n" equations for the system, one must substitute s in the characteristic equation in terms of ζ and ω_n, that is by:

$$s = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2} \qquad \text{(See figure 2-5).}$$

Therefore, it follows (Siljak) that

$$s^k = \omega_n^k \left[T_k(-\zeta) + j\sqrt{1 - \zeta^2}\, U_k(-\zeta) \right]$$

with $T_k(-\zeta)$ and $U_k(-\zeta)$ being Chebyshev functions of the first and second kind defined by

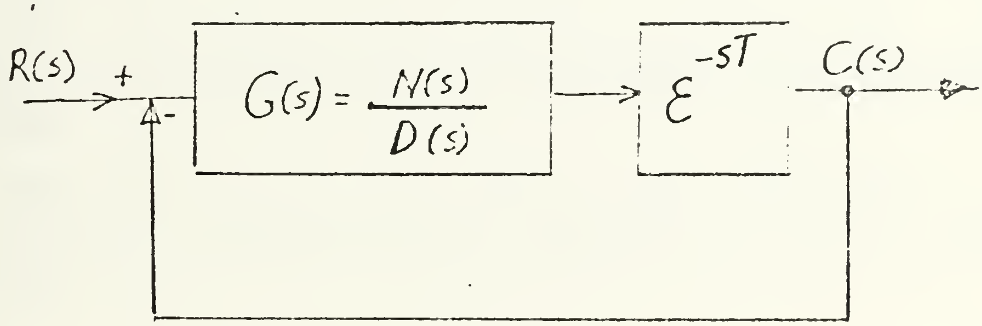
$$T_k(\zeta) = \cos(K\cos^{-1}\zeta) \quad \text{and} \quad U_k(\zeta) = \frac{\sin(K\cos^{-1}\zeta)}{\sin(\cos^{-1}\zeta)}$$

(see section I-B for the recursion relations), and note that

$$T_k(-\zeta) = (-1)^k T_k(\zeta) \quad \text{and} \quad U_k(-\zeta) = (-1)^{k+1} U_k(\zeta).$$

BLOCK DIAGRAM FOR A GENERAL SYSTEM WITH TIME DELAY

(a) UNITY FEED-BACK



(b) NON-UNITY FEED-BACK

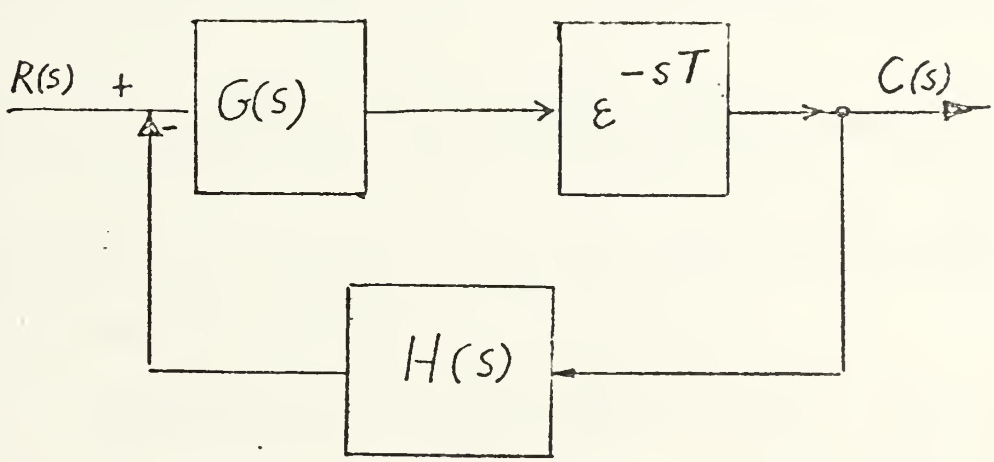


FIGURE 2-4 (a and b)

3. After the substitution for s in equation (II-6) is made, and by using the information in the previous paragraph and equation (II-7), the following result is obtained after separating reals from imaginaries:

$$\left[AB_1(\omega_n,\zeta) + BC_1(\omega_n,\zeta) + D_1(\omega_n,\zeta)\right] +$$
$$j\left[AB_2(\omega_n,\zeta) + BC_2(\omega_n,\zeta) + D_2(\omega_n,\zeta)\right] = 0 \qquad \text{(II-8)}$$

where

$$B_1 = \sum_{k=0}^{n} \omega_n{}^k \left[(-1)^k T_k(\zeta) b_k + \epsilon^{-\phi} c_k\left((-1)^k T_k(\zeta)\cos\theta - (-1)^{k+1}\sqrt{1-\zeta^2}\; U_k(\zeta)\sin\theta\right)\right]$$

$$C_1 = \sum_{k=0}^{n} \omega_n{}^k \left[(-1)^k T_k(\zeta) d_k + \epsilon^{-\phi} e_k\left((-1)^k T_k(\zeta)\cos\theta - (-1)^{k+1}\sqrt{1-\zeta^2}\; U_k(\zeta)\sin\theta\right)\right]$$

$$D_1 = \sum_{k=0}^{n} \omega_n{}^k \left[(-1)^k T_k(\zeta) f_k + \epsilon^{-\phi} g_k\left((-1)^k T_k(\zeta)\cos\theta - (-1)^{k+1}\sqrt{1-\zeta^2}\; U_k(\zeta)\sin\theta\right)\right]$$

and

$$B_2 = \sum_{k=0}^{n} \omega_n{}^k \left[(-1)^{k+1} U_k(\zeta)\sqrt{1-\zeta^2}\, b_k + \epsilon^{-\phi} c_k\left((-1)^{k+1} U_k(\zeta)\sqrt{1-\zeta^2}\cos\theta + (-1)^k T_k(\zeta)\;\sin\theta\right)\right]$$

$$C_2 = \sum_{k=0}^{n} \omega_n{}^k \left[(-1)^{k+1} U_k(\zeta)\sqrt{1-\zeta^2}\, d_k + \epsilon^{-\phi} e_k\left((-1)^{k+1} U_k(\zeta)\sqrt{1-\zeta^2}\cos\theta + (-1)^k T_k(\zeta)\;\sin\theta\right)\right]$$

$$D_2 = \sum_{k=0}^{n} \omega_n{}^k \left[(-1)^{k+1} U_k(\zeta)\sqrt{1-\zeta^2}\, f_k + \epsilon^{-\phi} g_k\left((-1)^{k+1} U_k(\zeta)\sqrt{1-\zeta^2}\cos\theta + (-1)^k T_k(\zeta)\;\sin\theta\right)\right]$$

{Note the identity $\epsilon^{sT} = \epsilon^{-\phi}(\cos\theta + j\sin\theta)$ for $\phi = \zeta\omega_n T$ and $\theta = \omega_n\sqrt{1-\zeta^2}\;T$}

4. The solution for A and B easily follows to be:

$$A = \frac{C_1D_2 - D_1C_2}{\triangle} \;;\quad B = \frac{D_1B_2 - D_2B_1}{\triangle} \;;\quad \triangle = B_1C_2 - C_1B_2$$

5. This derivation is the one followed by L. Eisenberg [4]. A derivation of parameter plane equations for a system with time delay and two variable parameters included in non-linear combinations, will follow.

6. System with the Product AB Present in its Characteristic Equation.

Consider the system in figure 2-4b with closed loop gain

$$\frac{C}{R}(s) = \frac{G(s)}{e^{Ts} + G(s)H(s)}$$

For

$$G(s) = \frac{G_N(s)}{G_D(s)} \quad \text{and} \quad H(s) = \frac{H_N(s)}{H_D(s)}$$

the characteristic equation will be:

$$F(s) = G_D(s)H_D(s)\epsilon^{Ts} + G_N(s)H_N(s) = \sum_{k=0}^{N} a_k s^k = 0 \qquad \text{(II-9)}$$

where:

$$a_k = A(b_k + c_k\epsilon^{Ts}) + B(d_k + e_k\epsilon^{Ts}) + AB(h_k + i_k\epsilon^{Ts}) + (f_k + g_k\epsilon^{Ts})$$

An expression for ϵ^{Ts} can be derived by using

$$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

and calling

$$\zeta\omega_n T = \phi \quad \text{and} \quad \omega_n\sqrt{1-\zeta^2}\, T = \theta .$$

So

$$\epsilon^{Ts} = \epsilon^{-\phi}\epsilon^{j\theta} = \epsilon^{-\phi}(\cos\theta + j\sin\theta) .$$

The expression for s^k as a function of ω_n, ζ, and the Chebyshev functions are also available. So the characteristic equation can be expressed as a function of ζ and ω_n as follows:

$$F(\zeta,\omega_n) = \sum_{k=0}^{N} a_k(-\omega_n)^k \left[T_k(\zeta) - j\sqrt{1-\zeta^2}\, U_k(\zeta)\right] \qquad \text{(II-10)}$$

where

$$a_k = A(b_k + c_k\epsilon^{-\phi}\cos\theta) + B(d_k + e_k\epsilon^{-\phi}\cos\theta) + (f_k + g_k\epsilon^{-\phi}\cos\theta)$$
$$+ AB(h_k + i_k\epsilon^{-\phi}\cos\theta) + j\epsilon^{-\phi}\sin\theta\,(Ac_k + Be_k + g_k + ABi_k) \equiv a_R + ja_I$$

Next, by defining $\beta_R = T_k(\zeta)$ and $\beta_I = \sqrt{1-\zeta^2}\ U_k(\zeta)$, the characteristic equation can be expressed as:

$$F(\zeta,\omega_n) = \sum_{k=0}^{n}(-\omega_n)^k \left[\alpha_R\beta_R + \alpha_I\beta_I + j(\alpha_I\beta_R - \alpha_R\beta_I)\right] = 0$$

which after separating into reals and imaginaries leads to:

$$\mathrm{Re}\left[F(\zeta,\omega_n)\right] = \sum_{k=0}^{N}(-\omega_n)^k(\alpha_R\beta_R + \alpha_I\beta_I) = 0$$

and ,

$$\mathrm{Im}\left[F(\zeta,\omega_n)\right] = \sum_{k=0}^{N}(-\omega_n)^k(\alpha_I\beta_R - \alpha_R\beta_I) = 0$$

And finally, by expanding terms inside parenthesis, this set of equations can take the form:

$$AB_1 + BC_1 + D_1 + ABE_1 = 0$$

$$AB_2 + BC_2 + D_2 + ABE_2 = 0$$

where:

$$B_1 = \sum_{k=0}^{N}(-\omega_n)^k\left[b_k T_k(\zeta) + c_k \varepsilon^{-\phi}\left(T_k(\zeta)\cos\theta + U_k(\zeta)\sqrt{1-\zeta^2}\ \sin\theta\right)\right]$$

$$C_1 = \sum_{k=0}^{N}(-\omega_n)^k\left[d_k T_k(\zeta) + e_k \varepsilon^{-\phi}\left(T_k(\zeta)\cos\theta + U_k(\zeta)\sqrt{1-\zeta^2}\ \sin\theta\right)\right]$$

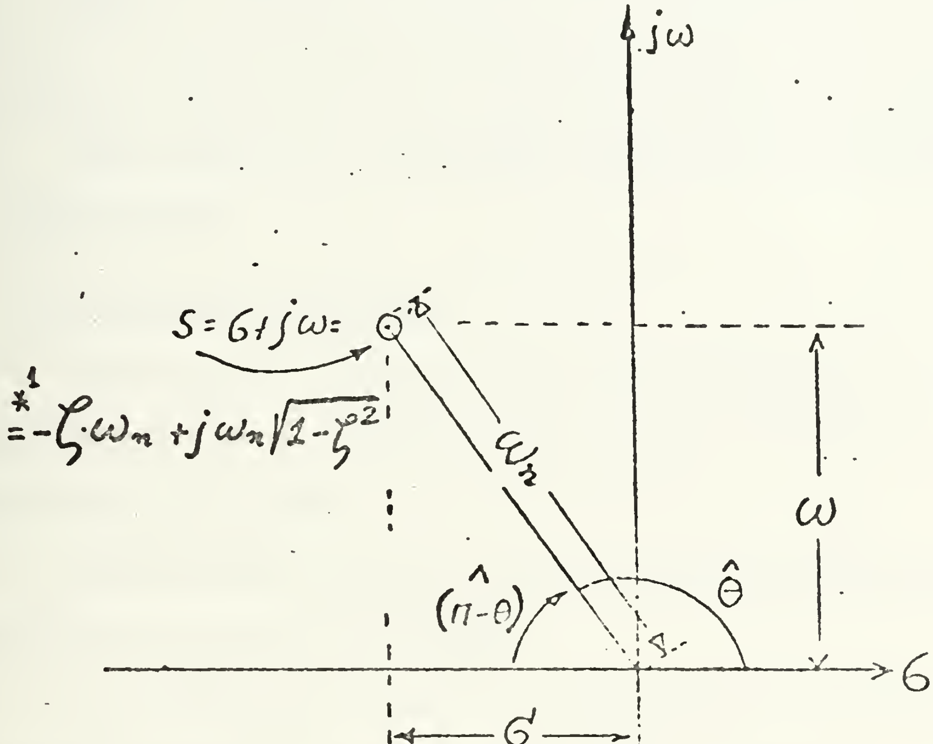
$$D_1 = \sum_{k=0}^{N}(-\omega_n)^k\left[f_k T_k(\zeta) + g_k \varepsilon^{-\phi}\left(T_k(\zeta)\cos\theta + U_k(\zeta)\sqrt{1-\zeta^2}\ \sin\theta\right)\right]$$

$$E_1 = \sum_{k=0}^{N}(-\omega_n)^k\left[h_k T_k(\zeta) + i_k \varepsilon^{-\phi}\left(T_k(\zeta)\cos\theta + U_k(\zeta)\sqrt{1-\zeta^2}\ \sin\theta\right)\right]$$

$$B_2 = \sum_{k=0}^{N}(-\omega_n)^k\left[-b_k U_k(\zeta)\sqrt{1-\zeta^2} + c_k \varepsilon^{-\phi}\left(T_k(\zeta)\sin\theta - U_k(\zeta)\sqrt{1-\zeta^2}\ \cos\theta\right)\right]$$

$$C_2 = \sum_{k=0}^{N}(-\omega_n)^k\left[-d_k U_k(\zeta)\sqrt{1-\zeta^2} + e_k \varepsilon^{-\phi}\left(T_k(\zeta)\sin\theta - U_k(\zeta)\sqrt{1-\zeta^2}\ \cos\theta\right)\right]$$

GEOMETRIC RELATIONS IN THE "S"-PLANE



Note:*1

$$\zeta = \left|\frac{\sigma}{\omega_n}\right|$$

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

FIGURE 2-5

$$D_2 = \sum_{k=0}^{N}(-\omega_n)^k\left[-f_k U_k(\zeta)\sqrt{1-\zeta^2} + g_k \epsilon^{-\phi}\left(T_k(\zeta)\sin\theta - U_k(\zeta)\sqrt{1-\zeta^2}\cos\theta\right)\right]$$

$$E_2 = \sum_{k=0}^{N}(-\omega_n)^k\left[-h_k U_k(\zeta)\sqrt{1-\zeta^2} + i_k \epsilon^{-\phi}\left(T_k(\zeta)\sin\theta - U_k(\zeta)\sqrt{1-\zeta^2}\cos\theta\right)\right]$$

The selection then of (II-11) is a matter of algebraic manipulation.

7. The derivation in paragraph 6 above is the one followed by S.E. Lamberski [5].

C. THE OBJECTIVES OF THIS WORK

1. What has been discussed up to this section was a review for people familiar with control theory and hopefully an introduction to the state of the art to others.

2. The sections to follow contain material associated with the fulfillment of objectives as follows:

a. Theoretical study of general systems with three variable parameters and with time delay. This is shown in Section V.

b. Study of actual engineering problems involving two or three variable parameters in systems with time delay.

c. (1) Since for such an engineering study to be complete, a stability analysis is needed and hence Section IV is devoted to presenting the state of the art.

(2) In Section III parameter plane equations are derived for a general system with two variable parameters and time delay. The reason that this derivation has been included is that it shows an interesting way to deal with the time delay difficulty. In all of its other aspects it is the same as the methods mentioned in this section, since they involve the already familiar algebraic manipulation of Siljak.

III. SYSTEMS WITH TWO VARIABLE PARAMETERS AND TIME DELAY

The purpose of this section is to derive parameter plane equations for a general system with time delay and two variable parameters. The basic parameter plane method is used, but it is different in the manner the time delay effect is treated. This approach will require a shorter computer program to implement.

Since the investigation of this kind of a system started with the analysis of a specific actual problem, equations for that kind of a problem are first derived. Extension to the general case is then covered in two further steps, by:

first, considering other possible real life system configurations and systematically filling in possible terms for a more general representation of all of them by one equation, which is true under certain conditions,

and second, removing the imposed conditions and thus getting the general parameter plane equation for a system with two variable parameters and time delay.

A. A SYSTEM WITH TIME DELAY IN THE FORWARD PATH, AND THE VARIABLE PARAMETERS IN THE FEEDBACK.

In this section, parameter plane equations for a system with time delay in the forward path and the variable parameters A,B in the feedback path, will be derived.

1. The System's Block Diagram

Assume that the system under consideration has a block diagram as in figure 3-1.

BLOCK DIAGRAM "ONE"

(NO PRODUCTS)

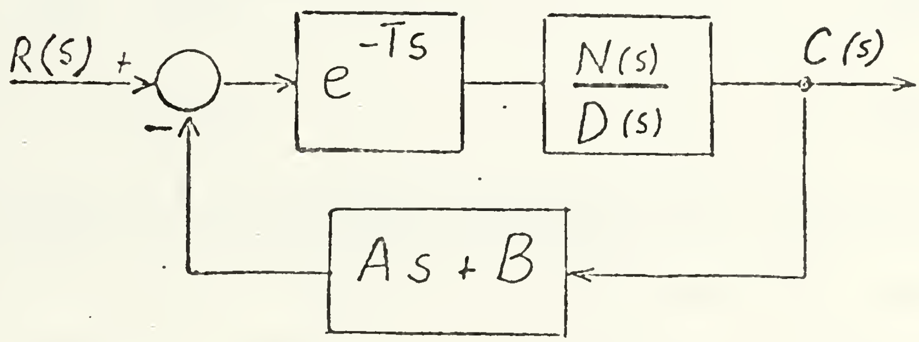


FIGURE 3-1

It turns out that the transfer function C/R is:

$$\frac{C}{R} = \frac{e^{-Ts}N(s)}{D(s) + e^{-Ts}(As + B)N(s)} \tag{III-1}$$

Therefore, let $As + B = AN_1(s) + BN_2(s)$ and assume that the coefficiencts of D(s) are $[d_k]_{k=0}^{n}$, of $N_2(s)$ are $[c_k]_{k=0}^{\ell}$, and of $N_1(s)$ are $[b_k]_{k=0}^{\ell}$, where n is the order of D(s) and ℓ is the order of N(s), increased by one, as $N_1(s) = s \cdot N(s)$.

2. Characteristic Equation

It follows from the above discussion that the system's characteristic equation can be written in the following form:

$$\sum_{k=0}^{n} d_k s^k + \sum_{k=0}^{\ell} a''_k e^{-Ts} s^k = 0 \tag{III-2}$$

where

$$a''_k = Ab_k + Bc_k \tag{III-3}$$

3. Derivation of a Set of Equations That Can Provide Constant "ω_n" or Constant "ζ" Curves in the "A,B" Plane.

Starting now from equation (III-2) one can express the complex quantity "s" in terms of ω_n and ζ and then attempt to get two equations of the form:

$$A \cdot B_1(\omega_n,\zeta) + B \cdot C_1(\omega_n,\zeta) + D_1(\omega_n,\zeta) = 0 \tag{III-4}$$

$$A \cdot B_2(\omega_n,\zeta) + B \cdot C_2(\omega_n,\zeta) + D_2(\omega_n,\zeta) = 0$$

One can see that the solution follows Siljak's basic steps. In this case, then, a solution for A and B is possible, for a given (ω_n,ζ) pair. Easily then the constant "ω_n" and the constant "ζ" curves will follow by keeping one of the two variables constant, and varying the other over a desired range of values.

Shown in Appendix B is a way a deriving equations in the form of the ones in (III-4). In fact one gets:

$$A\left[e^{T\omega_n\zeta}\sum_{k=0}^{\ell} b_k\omega_n^k X_k(\zeta,\omega_n)\right] + B\left[e^{T\omega_n\zeta}\sum_{k=0}^{\ell} c_k\omega_n^k X_k(\zeta,\omega_n)\right] + \left[\sum_{k=0}^{n} d_k\omega_n^k T_k(\zeta)\right] = 0 \tag{III-5a}$$

$$A\left[e^{T\omega_n\zeta}\sum_{k=0}^{\ell} b_k\omega_n^k\left(\zeta\cdot X_k(\zeta,\omega_n) + X_{k-1}(\zeta,\omega_n)\right)\right] +$$

$$B\left[e^{T\omega_n\zeta}\sum_{k=0}^{\ell} c_k\omega_n^k\left(\zeta\cdot X_k(\zeta,\omega_n) + X_{k-1}(\zeta,\omega_n)\right)\right] + \tag{III-5b}$$

$$\left[\sum_{k=0}^{n} d_k\omega_n^k\left(\zeta\cdot T_k(\zeta) + T_{k-1}(\zeta)\right)\right] = 0$$

where α_k, β_k, c_k are constant coefficients of the characteristic equation of the system as mentioned before, and T_{k+1}, x_{k+1} are given from the following recurrence relations:

$$\begin{aligned} T_{k+1} &= -2\zeta\cdot T_k - T_{k-1} \\ X_{k+1} &= -2\zeta\cdot X_k - X_{k-1} \end{aligned} \tag{III-6}$$

and subject to the following initial conditions:

$$X_{-1} = -\left(\sqrt{1-\zeta^2}\sin\left[T\omega_n\sqrt{1-\zeta^2}\right] + \zeta\cos\left[T\omega_n\sqrt{1-\zeta^2}\right]\right)$$

$$X_o = \cos\left[T\omega_n\sqrt{1-\zeta^2}\right]$$

$$X_1 = \left(\sqrt{1-\zeta^2}\sin\left[T\omega_n\sqrt{1-\zeta^2}\right] - \zeta\cos\left[T\omega_n\sqrt{1-\zeta^2}\right]\right)$$

$$T_{-1} = -\zeta$$

$$T_o = 1$$

$$T_1 = -\zeta \tag{III-7}$$

By comparing equations (III-4) and (III-5),

$$B_1 = e^{T\omega_n\zeta}\sum_{k=0}^{\ell} b_k\omega_n^k X_k(\zeta,\omega_n)$$

and the rest of the values for $[B_i, C_i, D_i]_{i=1,2}$ follow easily. So if $B_1C_2 - B_2C_1 \neq 0$, from equation (III-4) follows:

$$A = (C_1D_2 - C_2D_1) \div (B_1C_2 - B_2C_1)$$
$$B = (B_2D_1 - D_2B_1) \div (B_1C_2 - B_2C_1) \qquad \text{(III-8)}$$

4. <u>Derivation of an Equation That Can Provide Constant "σ" Curves.</u>

The real roots of equation (III-2) may be found by substituting $s = \sigma$, the required root. It follows then:

$$\sum_{k=0}^{n} d_k \sigma^k + \sum_{k=0}^{\ell} e^{-T\sigma} a_k'' \sigma^k = 0$$

where $$a_k'' = A \cdot b_k + B \cdot c_k \qquad \text{(III-3)}$$

or

$$\sum_{k=0}^{n} d_k \sigma^k + \sum_{k=0}^{\ell} e^{-T\sigma} Ab_k \sigma^k + \sum_{k=0}^{\ell} e^{-T\sigma} Bc_k \sigma^k = 0$$

and after bringing A,B, and $e^{-T\sigma}$ outside of the summation operator, we get finally:

$$A\left[e^{-T\sigma} \sum_{k=0}^{\ell} b_k \sigma^k\right] + B\left[e^{-T\sigma} \sum_{k=0}^{\ell} c_k \sigma^k\right] + \sum_{k=0}^{n} d_k \sigma^k = 0 \qquad \text{(III-9)}$$

5. <u>Computer Program</u>

In Appendix D, a computer program able to draw constant σ, ζ and ω_n curves or combinations of them is listed. Computations take place as follows (Notice that reference is made to equations of Appendix B):

a. The Case of Constant "ζ" and "ω_n" Curves

(1) B_1, C_1, D_1, B_2, C_2, and D_2 are calculated from Equations (B-15) for the constant quantity of the curve wanted. For each value of the running variable, ω_n^k, X_k, X_{k-1}, T_k and T_{k-1} are calculated by iteration from the appropriate recursion and initial value formulas

as mentioned before and multiplied by the proper b_k, c_k and d_k as necessary. The quantity $e^{T\omega_n \zeta}$ is also calculated to form $[B_i, C_i]_{i=1,2}$.

(2) By using equations (B-16), a point P(A,B) is drawn on the A,B plane for the constant value curve in discussion, whenever the running variable takes one of the values we specify it to span.

b. The Case of Constant "σ" Curves

As one can see, equation (III-9) represents a straight line in the A,B plane for given "σ". So in the computer program of Appendix D, the quantities,

$$e^{-T\sigma};\ \sum_{k=0}^{\ell} b_k \sigma^k;\ \sum_{k=0}^{\ell} c_k \sigma^k \text{ and } \sum_{k=0}^{n} d_k \sigma^k$$

are calculated for selected "σ", and constant σ lines are plotted by using appropriate intercepts.

c. Basic Inputs

In addition to the types of curves, the selected values needed, the span of the running variable and some other information necessary to the drawing unit, one must input the order of the characteristic equation (which coincides with the order of D(s) for the most usual case when order D(s) > N(s)), the maximum degree of the exponent in "s" that the time delay term, (e^{-Ts}), appears in the characteristic equation, the time delay constant "T" and the coefficients of the system b_k, c_k, and d_k.

B. OTHER PRACTICAL CONFIGURATIONS - GENERALIZATION

1. Other Possible Time-Delay and Plant Configurations for Feed-Back Compensation (A and B Shown in the Feed-Back Loop)

In order to get a generalization of the parameter plane equations for a system with time delay, one should devise possible configurations

of the plant, the compensation unit and the time delay and after solving for the parameter plane equations for each case, should then try to for-see how these equations would look for the general system with time delay.

a. In Figure 3-2 the first of a set of three configurations is shown. But noting that the loop gain of this system is exactly the same as for the system discussed in section IIIA, one shouldn't expect any differences in the characteristic equations of these two systems. Therefore, the characteristic equation is $D(s) + N(s)(As + B)e^{-Ts} = 0$.

b. The next system to consider is the one shown in Figure 3-3. Since for this system:

$$\frac{C}{R} = \frac{\dfrac{N(s)}{D(s)}}{1 + \dfrac{N(s)}{D(s)}\left[Ase^{-Ts} + B\right]}$$

it follows that the characteristic equation for this system will be:

$$D(s) + B\,N(s) + e^{-Ts}As\,N(s) = 0$$

So the basic difference with the original system of section IIIA, exists in the fact that the term multiplying the variable B <u>is N(s)</u> for the latter case <u>instead</u> of $\underline{N(s)\cdot e^{-Ts}}$ for the former. As a consequence of that and going over the procedure of section IIIA, one can see that no $e^{-T\omega_n\zeta}$ term appears in the set of equations (III-5) as far as C_1 and C_2 are concerned. Furthermore, only T_k, U_k terms will show up in the solution for C_1 and C_2. As far as B_1, B_2, and D_1, D_2 are concerned, there will be no change from the solution of section IIIA.

c. The last system to be considered for the feed-back compensation case with A and B in the feed-back loop, is the one shown in Figure 3-4. In this case:

$$\frac{C}{R} = \frac{N(s)}{D(s) + N(s)(As + Be^{-Ts})}$$

BLOCK DIAGRAM "TWO"

(NO PRODUCTS)

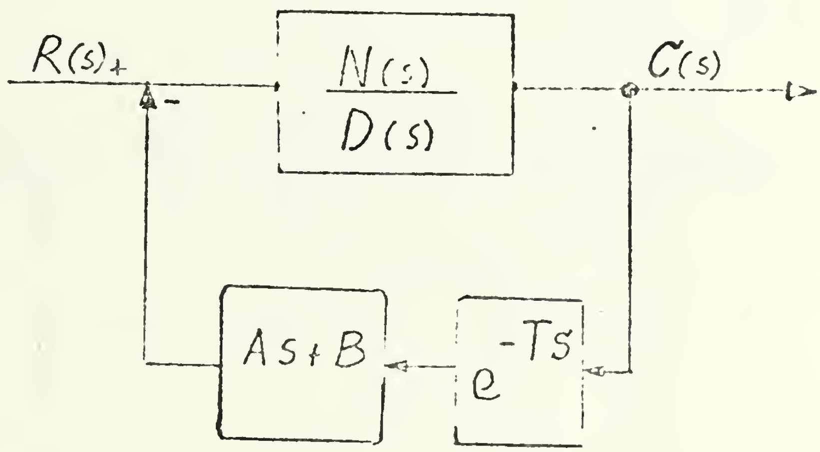


FIGURE 3-2

BLOCK DIAGRAM "THREE"

(NO PRODUCTS)

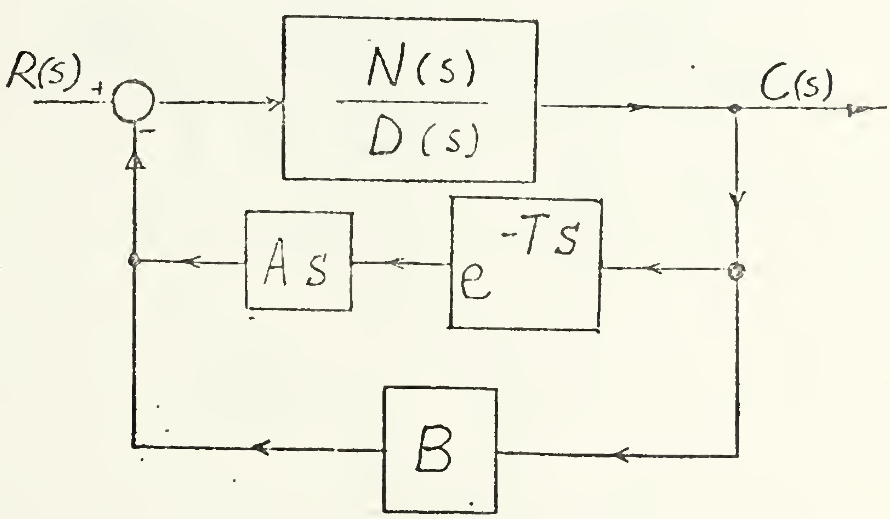


FIGURE 3-3

BLOCK DIAGRAM "FOUR"

(NO PRODUCTS)

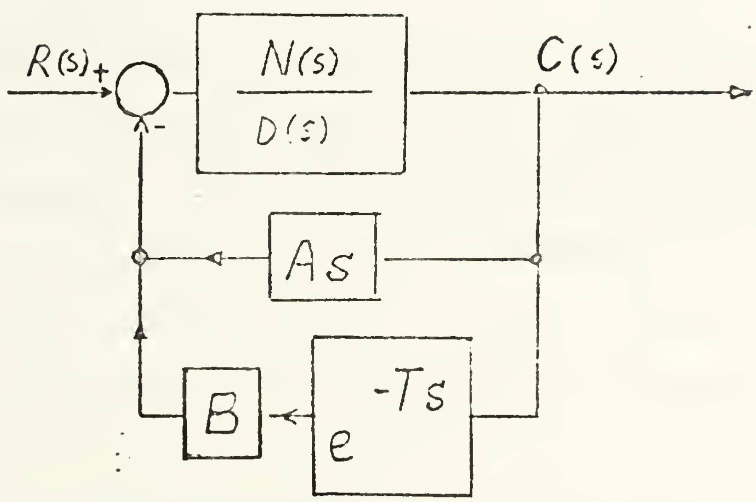


FIGURE 3-4

Therefore, the characteristic equation is:

$$D(s) + AsN(s) + e^{-Ts}BN(s) = 0$$

Again by comparison with the characteristic equation of the system discussed in Section IIIA, one can see that the difference this time exists in the term multiplying the "A" variable parameter. In fact, instead of $\underline{A(s\cdot N(s)\cdot e^{-Ts})}$ it is $\underline{A(s\cdot N(s))}$. So following again the procedure in Section IIIA, one will get no $e^{-T\omega_n\zeta}$ term appearing in the set of equations (III-5) as far as B_1 and B_2 are concerned this time. Furthermore, only T_k, U_k terms will show up in the solution for B_1 and B_2. The solutions for C_1, C_2, and D_1, D_2 will remain the same as for the system of Section IIIA.

d. Conclusion

The solutions for the parameter plane equations for the systems described in paragraphs IIIB1b, IIIB1c, remained essentially the same as for the system in section IIIA, with the only difference of minor changes on the C_i or B_i coefficients $(i = 1,2)$, whenever the time delay factor doesn't show in the characteristic equation to multiply one of the two variable parameters A, B. These results are tabulated in Table 3 - 1.

2. <u>Possible Time Delay and Plant Configurations for the Case of a Lead-Lag Compensation on a Fixed Feed-Back System.</u>

a. Consider first the system as in Figure 3-5. The transfer function for this system is:

$$\frac{C}{R} = \frac{\frac{(s+A)}{(s+B)}\cdot\frac{N(s)}{D(s)}\cdot e^{-Ts}}{1 + k\frac{(s+A)}{(s+B)}\cdot\frac{N(s)}{D(s)}\cdot e^{-Ts}} =$$

$$= \frac{(s+A)\cdot N(s)\cdot e^{-Ts}}{\underbrace{sD(s)}_{d_k} + e^{-Ts}\underbrace{k\cdot s\cdot N(s)}_{d_k'} + \underbrace{BD(s)}_{c_k} + e^{-Ts}A\underbrace{(K\cdot N(s))}_{b_k}} \qquad \text{(III-10)}$$

TABLE 3-1

SYSTEM ↓	"CHARACTERIST. EQUATION" ↓	B_1 ↓	C_1 ↓	D_1 ↓	B_2 ↓	C_2 ↓	D_2 ↓
OF FIGURE 3-3	$D(s)+B\cdot N(s)+e^{-Ts}A\cdot s\cdot N(s)=0$	As for the system of section III A..	$\sum_{k=0}^{\ell}(c_k\cdot\omega_n^k\cdot T_k)$	As for the system of section III A	As for the system of section III A	$\sum_{k=0}^{\ell}c_k\cdot\omega_n^k\cdot(\xi T_k+T_{k-1})$	As for the system of section III A
OF FIGURE 3-4	$D(s)+A\cdot s\cdot N(s)+e^{-Ts}\cdot B\cdot N(s)=0$	$\sum_{k=0}^{\ell}(b_k\cdot\omega_n^k\cdot T_k)$	As for the system of section III A..	—11—	$\sum_{k=0}^{\ell}b_k\cdot\omega_n^k\cdot(\xi T_k+T_{k-1})$	As for the system of section III A	—11—
OF FIGURE 3-1 (SECTION III A).	$D(s)+A\cdot s\cdot N(s)\cdot e^{-Ts}+e^{-Ts}\cdot B\cdot N(s)=0$	$e^{T\omega_n\xi}\cdot\sum_{k=0}^{\ell}(b_k\cdot\omega_n^k\cdot X_k)$	$e^{T\omega_n\xi}\cdot\sum_{k=0}^{\ell}(c_k\cdot\omega_n^k\cdot X_k)$	$\sum_{k=0}^{n}(d_k\cdot\omega_n^k\cdot T_k)$	$e^{T\cdot\omega_n\cdot\xi}\cdot\sum_{k=0}^{\ell}b_k\cdot\omega_n^k\cdot(\xi X_k+X_{k-1})$	$e^{T\cdot\omega_n\cdot\xi}\cdot\sum_{k=0}^{\ell}c_k\,\omega_n^k\cdot(\xi X_k+X_{k-1})$	$\sum_{k=0}^{n}d_k\cdot\omega_n^k\cdot(\xi\cdot T_k+T_{k-1})$.-

BLOCK DIAGRAM "FIVE"

(NO PRODUCTS)

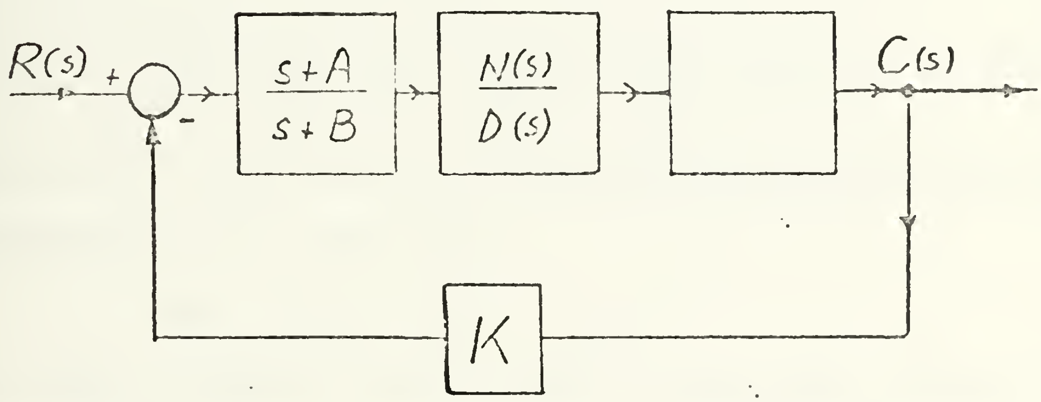


FIGURE 3-5

By examining closer the characteristic equation of this system the following facts should be noted:

(1) As far as the terms multiplying the A and B variable parameters are concerned the situation is similar to the one examined in paragraph IIIB1b with the system shown in Figure 3-3.

(2) For the first time, terms independent of A and B and of the form "e^{-Ts} x(polynomial in s)" are encountered.

b. In the case where the time delay appears in the feed-back path with all the other components of the system being as shown in Figure 3-5, the loop gain, and so the characteristic equation, remain unchanged.

c. The characteristic equation of the system in Figure 3-5 can be formulated in a more elegant way as follows:

$$\sum_{k=0}^{n} d_k s^k + \sum_{k=0}^{n} e^{-Ts} d_k' s^k + B \sum_{k=0}^{n} c_k s^k + A \sum_{k=0}^{n} e^{-Ts} b_k s^k = 0 \qquad \text{(III-11)}$$

where d_k, d_k', c_k, and b_k are the constant terms encountered in the following polynomials in "s", respectively:

sD(s), ksN(s), D(s), and kN(s).

The fact that the summation operators shown in the equation (III-11) are all operating in a range from "0" to "n" is perfectly legitimate and what is meant by that is illustrated as follows and in relation to the system of Figure 3-5.

(1) Consider: k =1 and $\frac{N(s)}{D(s)} = \frac{2s + 3}{4s^3 + 5s^2 + 6s + 7}$

where order D(s) > order N(s), as usually happens with most plants.

(2) It follows that:

(a) $sD(s) = 4s^4 + 5s^3 + 6s^2 + 7s^1 + 0s^0$

(b) $ksN(s) = 2s^2 + 3s^1 + 0s^0$

(c) $D(s) = 4s^3 + 5s^2 + 6s^1 + 7s^0$

(d) $kN(s) = 2s^1 + 3s^0,$

where polynomial of max(order) = 4, is sD(s). But for the purpose of neater summation operations in a digital computer one could form the characteristic equation as implied by equation (III-11), as follows:

$$\sum_{k=0}^{n=4} d_k s^k + \sum_{k=0}^{n=4} e^{-Ts} d'_k s^k + B \sum_{k=0}^{n=4} c_k s^k + A \sum_{k=0}^{n=4} e^{-Ts} b_k s^k = 0 \qquad \text{(III-11a)}$$

where d_k, d'_k, c_k, b_k for the system are tabulated in Table 3-2, and n=4 is implied by the fact that the polynomial of maximum order for this case is of order 4.

TABLE 3-2

Polynomial	k = 4	3	2	1	0
sD(s)	$d_4 = 4$	$d_3 = 5$	$d_2 = 6$	$d_1 = 7$	$d_o = 0$
k sN(s)	$d'_4 = 0$	$d'_3 = 0$	$d'_2 = 2$	$d'_1 = 3$	$d'_o = 0$
D(s)	$c_4 = 0$	$c_3 = 4$	$c_2 = 5$	$c_1 = 6$	$c_o = 7$
kN(s)	$b_4 = 0$	$b_3 = 0$	$b_2 = 0$	$b_1 = 2$	$b_o = 3$

3. Generalization

a. Having in mind the form of equation (III-11) as well as the results tabulated in Table 3-1, one can deduce that a more general form can be given to the characteristic equations of all systems discussed up to now in section III and this is as follows:

$$\sum_{k=0}^{n} d_k s^k + \sum_{k=0}^{n} e^{-\delta_{d2} \cdot T \cdot s} d'_k s^k + B \sum_{k=0}^{n} e^{-\delta_c \cdot T \cdot s} c_k s^k + A \sum_{k=0}^{n} e^{-\delta_b \cdot T \cdot s} b_k s^k = 0 \qquad \text{(III-12)}$$

where d_k, d'_k, c_k, and b_k are defined as in the previous illustrative example of this section (p.), and δ_{d2}, δ_c, δ_b are variables that take on values of 0 or 1 depending on the system under examination.

b. Constant "σ" Curves

For s = 6, one gets by direct substitution from (III-12) the following equation representing as usual, a family of straight lines:

$$A\left[\sum_{k=0}^{n} e^{-\delta_b \cdot T \cdot \sigma} b_k \sigma^k\right] + B\left[\sum_{k=0}^{n} e^{-\delta_c \cdot T \cdot \sigma} c_k \sigma^k\right] + \left[\sum_{k=0}^{n} (d_k + e^{-\delta_{d2} \cdot T \cdot \sigma} d'_k)\sigma^k\right] = 0 \quad \text{(III-13)}$$

c. Constant "ζ" or Constant "ω_n" Curves

It has already been defined that:

$$\delta_{i\,(i=d_2,c,b)} = \begin{cases} 0 \\ \text{or} \\ 1 \end{cases} .$$

If now, δ'_j is defined as:

$$\delta'_{j\,(j=d_2,c,b)} = \begin{cases} 1 \\ \text{or} \\ 0 \end{cases} ,$$

it follows that:

$$\delta_i + \delta'_j \equiv 1 \quad \text{(III-14)}$$

So by using δ_i and δ'_j one can selectively maintain or discard the different terms that must or must not show up in the parameter plane solution of a particular system if this system has a characteristic equation of the form as in (III-12).

(1) Example. By examining the solutions for B_1 and B_2 in table 3-1, it follows that for the systems discussed up to now, B_1 takes one of the forms:

$$\text{(a)} \quad B_1 = \sum_{k=0}^{\ell} b_k \omega_n^k T_k$$

$$\text{or (b)} \quad B_1 = e^{T\omega_n \zeta} \sum_{k=0}^{\ell} b_k \omega_n^k X_k$$

This statement can be reduced into one single equation by using (III-14), and the fact that B_1 is related to b_k's and therefore to the equation $\delta_b + \delta'_b \equiv 1$ (observe δ_b to be related with "b_k" coefficients in (III-12).

(c) $B_1 = e^{-\delta_b \cdot T \cdot \omega_n \cdot \zeta} \cdot \delta_b \cdot \sum_{k=0}^{n} b_k \omega_n^{\ k} X_k + \delta_b' \sum_{k=0}^{n} b_k \omega_n^{\ k} T_k$ (III-15a)

(d) Similarly:

$$B_2 = e^{-\delta b \cdot T \cdot \omega_n \cdot \zeta} \cdot \delta_b \cdot \sum_{k=0}^{n} b_k \omega_n^{\ k} (\zeta X_k + X_{k-1})$$
$$+ \delta_b' \sum_{k=0}^{n} b_k \omega_n^{\ k} (\zeta T_k + T_{k-1}) \qquad \text{(III-15b)}$$

Solutions for C_1, C_2 and D_1, D_2 follow similar patterns.

(e) It is of interest to note that if $\delta_i = 1$ ($\delta_j'=0$), the presence of X_k's and X_{k-1}'s is implied in the relevant expression for B_1, B_2, C_1, C_2, or D_1, D_2. If in the contrary, $\delta_j' = 1$ ($\delta_i=0$), the presence of T_k's and T_{k-1}'s is implied.

(2) <u>Derivation of Equations</u>

For one to actually solve for the parameter plane equations, one must make use of the fact (as in section IIIA), that:

$$s = \omega_n(\cos\theta + j\sin\theta) = \omega_n e^{j\theta}$$

and apply it in equation (III-12) to get:

$$\sum_{k=0}^{n} d_k \omega_n^{\ k} (\cos k\theta + j \sin k\theta) + \sum_{k=0}^{n} d_k' \left(e^{-\delta_{d2} \cdot T \cdot (\cos\theta + j\sin\theta)\omega_n}\right) \omega_n^{\ k} \cdot e^{jk\theta} +$$
$$\sum_{k=0}^{n} \left(Be^{-\delta_c T(\cos\theta + j\sin\theta)\omega_n}\right) c_k \omega_n^{\ k} e^{jk\theta} +$$
$$\sum_{k=0}^{n} \left(Ae^{-\delta_b T(\cos\theta + j\sin\theta)\omega_n}\right) b_k \omega_n^{\ k} e^{jk\theta} = 0$$

which, with appropriate manipulation inside the summation operators involving d_k', c_k and b_k, yields the result shown in equation (III-16).

(See next page.)

$$\sum_{k=0}^{n} d_k \omega_n^k (\cos k\theta + j \sin k\theta) +$$

$$e^{\delta_{d2} \cdot T \cdot \omega_n \cdot \zeta} \sum_{k=0}^{n} d_k' \omega_n^k \left\{ \left[\delta_{d_2} \cdot \cos(k\theta + \omega_n \cdot T \cdot \zeta) + \delta_{d_2}' \cdot \cos(k\theta) \right] + j \left[\delta_{d_2} \cdot \sin(k\theta + \omega_n \cdot T \cdot \zeta) + \delta_{d_2}' \cdot \sin(k\theta) \right] \right\} +$$

$$Be^{\delta_c \cdot T \cdot \omega_n \cdot \zeta} \sum_{k=0}^{n} c_k \omega_n^k \left\{ \left[\delta_c \cdot \cos(k\theta + \omega_n \cdot T \cdot \zeta) + \delta_c' \cdot \cos(k\theta) \right] + j \left[\delta_c \cdot \sin(k\theta + \omega_n \cdot T \cdot \zeta) + \delta_c' \cdot \sin(k\theta) \right] \right\} +$$

$$Ae^{\delta_b \cdot T \cdot \omega_n \cdot \zeta} \sum_{k=0}^{n} b_k \omega_n^k \left\{ \left[\delta_b \cdot \cos(k\theta + \omega_n \cdot T \cdot \zeta) + \delta_b' \cdot \cos(k\theta) \right] + j \left[\delta_b \cdot \sin(k\theta + \omega_n \cdot T \cdot \zeta) + \delta_b' \cdot \sin(k\theta) \right] \right\} . \qquad \text{(III-16)}$$

From this stage by using appropriate substitutions for cosines and sines with arguments $(k\theta)$ or $(k\theta + \omega_n \cdot T \cdot \zeta)$, in terms of X_k's, Y_k's, T_k's, and U_k's and their recurrence relations, exactly as defined derived and used in section IIIA, one can form the following equations after separating reals from imaginaries:

$$\sum_{k=0}^{n} d_k \omega_n^k T_k + e^{\delta_{d2} \cdot T \cdot \omega_n \cdot \zeta} \sum_{k=0}^{n} d_k' \omega_n^k (\delta_{d2} \cdot X_k + \delta_{d2}' T_k) +$$

$$Be^{\delta_c \cdot T \cdot \omega_n \cdot \zeta} \sum_{k=0}^{n} c_k \omega_n^k (\delta_c X_k + \delta_c' T_k) +$$

$$Ae^{\delta_b \cdot T \cdot \omega_n \cdot \zeta} \sum_{k=0}^{n} b_k \omega_n^k (\delta_b X_k + \delta_b' T_k) = 0 \qquad \text{(III-17a)}$$

(see next page)

and:

$$\sum d_k \omega_n^{\ k} U_k + e^{\delta_{d2}\cdot T\cdot\omega_n\cdot\zeta} \sum d_k' \omega_n^{\ k}(\delta_{d_2} Y_k + \delta_{d_2}' U_k) +$$

$$Be^{\delta_c\cdot T\cdot\omega_n\cdot\zeta} \sum c_k \omega_n^{\ k}(\delta_c Y_k + \delta_c' U_k) +$$

$$Ae^{\delta_b\cdot T\cdot\omega_n\cdot\zeta} \sum b_k \omega_n^{\ k}(\delta_b Y_k + \delta_b' U_k) = 0 \qquad \text{(III-17b)}$$

Equation (III-17b) can be expressed in terms of $_k$'s and T_k's instead of U_k's and $_k$'s, in exactly the same manner as in section IIIA by using equations (B-9c) and (B-12c) of Appendix B. So the second equation of the parameter plane system becomes:

$$\sum_{k=0}^{n} d_k \omega_n^{\ k}(\zeta T_k + T_{k-1}) + e^{\delta_{d2}\cdot T\cdot\omega_n\cdot\zeta} \sum_{k=0}^{n} d_k' \omega_n^{\ k}\left[\delta_{d_2}(\zeta X_k + X_{k-1}) + (\delta_{d2}'(\zeta T_k + T_{k-1})\right] +$$

$$Be^{\delta_c\cdot T\cdot\omega_n\cdot\zeta} \sum_{k=0}^{n} c_k \omega_n^{\ k}\left[\delta_c(\zeta X_k + X_{k-1}) + \delta_c'(\zeta T_k + T_{k-1})\right] +$$

$$Ae^{\delta_b\cdot T\cdot\omega_n\cdot\zeta} \sum_{k=0}^{n} b_k \omega_n^{\ k}\left[\delta_b(\zeta X_k + X_{k-1}) + \delta_b'(\zeta T_k + T_{k-1})\right] = 0 \qquad \text{(III-17c)}$$

For easier reference, recurrence equation as well as special symbol relations are listed below:

(a) $T_{k+1} = -2\zeta T_k - T_{k-1}$

(b) $X_{k+1} = -2\zeta X_k - X_{k-1}$

(c) Initial condition relations usually necessary in digital computer calculations:

(i) $T_{-1} = -\zeta,\ T_o = 1,\ T_1 = -\zeta$

(ii) $X_{-1} = -\left[\sqrt{1-\zeta^2}\sin(T\omega_n\sqrt{1-\zeta^2}) + \zeta\cos(T\omega_n\sqrt{1-\zeta^2})\right]$

$X_o = \cos(T\omega_n\sqrt{1-\zeta^2})$

$$X_1 = \sqrt{1-\zeta^2}\sin(T\omega_n\sqrt{1-\zeta^2}) - \zeta\cos(T\omega_n\sqrt{1-\zeta^2})$$

(iii) $\delta'_j + \delta_i = 1$

$$\delta'_j = \begin{cases} 0 \\ 1 \end{cases}$$

$$\delta_i = \begin{cases} 1 \\ 0 \end{cases}$$

$$i = j = (d_2, c, b)$$

(3) Parameter Plane Curves. Constant "ω_n" and/or constant "ζ" curves can be drawn in the parameter plane in exactly the same manner as discussed in section I-B (Siljak's approach).

(4) Conclusions.

(a) By comparison of equations (III-17) and the solutions for $B_1, C_1, D_1, B_2, C_2, D_2$ tabulated in Table 3-1, one can observe that the derived equations (III-17) completely cover each case and furthermore they cover the case of the system shown in Figure 3-5.

(b) The equations (III-17), are valid for the case that the terms encountered in the characteristic equations examined, and involving the variable parameters A and B are either of the form:

$$(Ae^{-Ts})\cdot(\text{Polynomial}_1(s)) + B\cdot(\text{Polynomial}_2(s))$$

or of the form:

$$A\cdot(\text{Polynomial}_3(s)) + (Be^{-Ts})\cdot(\text{Polynomial}_4(s))$$

This was because of the specific features of the systems examined. But since the terms not involving A and B in equation (III-11) are of the form:

$$e^{-Ts}\cdot(\text{Polynomial}_5(s)) + \text{Polynomial}_6(s) ,$$

the possibility exists for a more general case with A,B as well as AB

appearing in the characteristic equation:

$$F(s) + F'(s)e^{-Ts} + A\left[B(s) + B'(s)e^{-Ts}\right] + B\left[C(s) + C'(s)e^{-Ts}\right] + AB\left[D(s) + D'(s)e^{-Ts}\right] = 0 \qquad \text{(III-18)}$$

The parameter plane analysis of such a general system follows next.

C. NON-LINEAR VARIABLE PARAMETER PRODUCTS INCLUDED IN THE CHARACTERISTIC EQUATION

1. In figure 3-6 a two variable parameter system is shown, which will result in a characteristic equation, containing the AB product of these two parameters, as follows:

$$F(s) = \left[D(s) + sN(s)e^{-Ts}\right] + A\left[N(s)e^{-Ts}\right] + B\left[s^2N(s)e^{-Ts}\right] + AB\left[sN(s)e^{-Ts}\right] = 0$$

or,

$$F(s) = \sum_{k=0}^{n+1}(f_k s^k + f'_k s^k e^{-Ts}) + A\sum_{k=0}^{n}(b'_k s^k e^{-Ts}) + B\sum_{k=0}^{n+2}(c'_k s^k e^{-Ts}) + AB\sum_{k=0}^{n+1}(d'_k s^k e^{-Ts}) = 0 \qquad \text{(III-19)}$$

2. However, in order to gain more generality so as to apply the results into a broader system classification, it is wise to solve for the parameter plane equations of the following characteristic equation:

$$\sum_{k=0}^{n}(f_k + f'_k e^{-Ts})s^k + \sum_{k=0}^{\ell}A(b_k + b'_k e^{-Ts})s^k + \sum_{k=0}^{m}B(c_k + c'_k e^{-Ts})s^k + \sum_{k=0}^{p}AB(d_k + d'_k e^{-Ts})s^k = 0 \qquad \text{(III-20)}$$

and by arbitrarily choosing $n = \max(n, \ell, m, p)$ as well as by filling non-existent coefficients of lower degree polynomials by zeros:

BLOCK DIAGRAM "SIX"

(PARAMETER "PRODUCTS" INCL.)

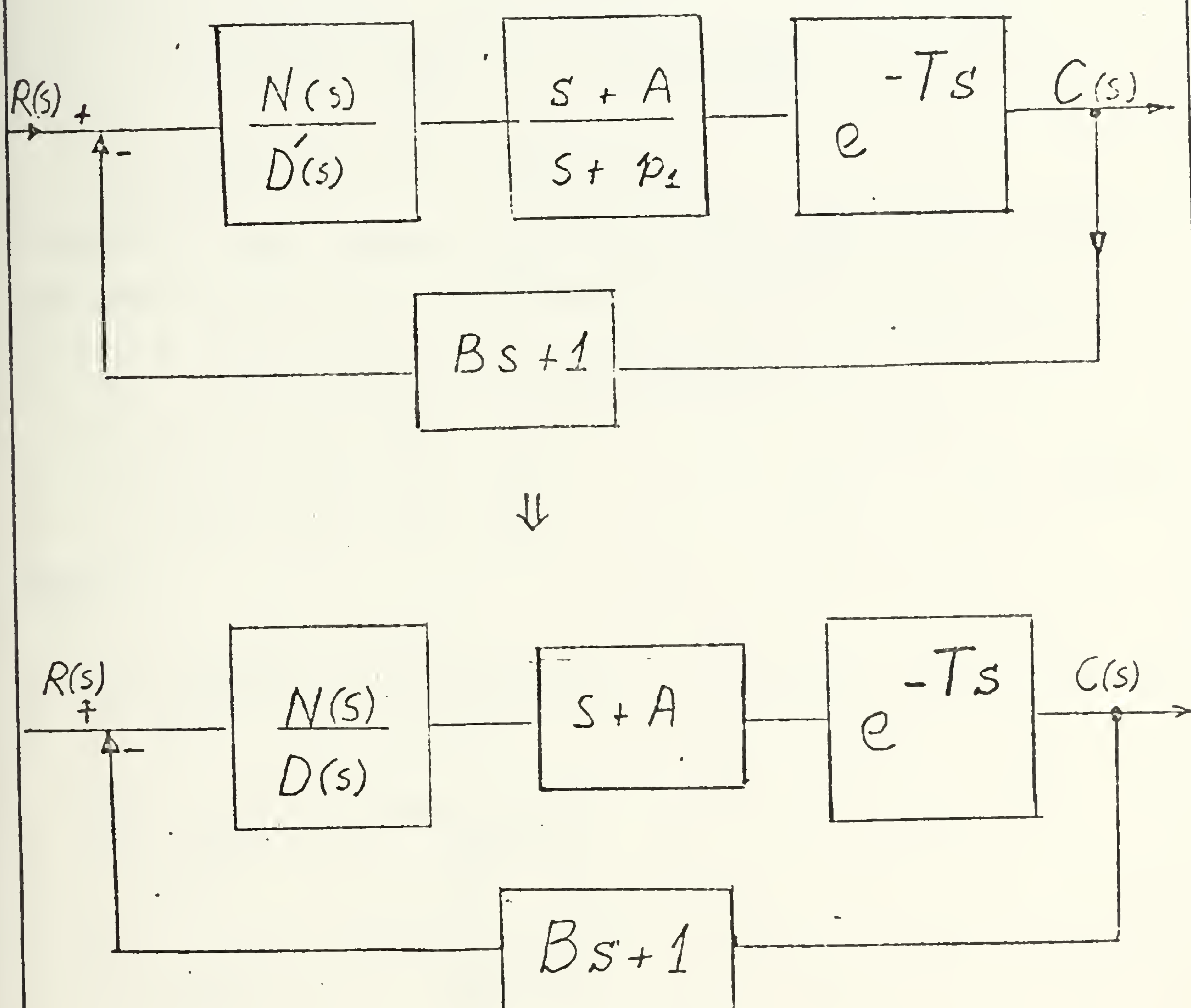


FIGURE 3-6

$$\sum_{k=0}^{n}(f_k + f_k' e^{-Ts})s^k + A\sum_{k=0}^{n}(b_k + b_k' e^{-Ts})s^k +$$

$$B\sum_{k=0}^{n}(c_k + c_k' e^{-Ts})s^k + AB\sum_{k=0}^{n}(d_k + d_k' e^{-Ts})s^k = 0$$

3. The solution for constant ω_n and ζ curves is again based on the equations:

$$s = \omega_n(\cos\theta + j\sin\theta) = \omega_n e^{j\theta}$$

and $$x_k = \cos(k\theta + \omega_n T\zeta), \quad y_k = \sin(k\theta + \omega_n T\zeta)$$

$$T_k = \cos k\theta \quad , \quad U_k = \sin k\theta$$

(escorted by their recursion relations), being substituted in equation (III-20), and thereafter on a separation of reals and imaginaries.

4. Actually, the set of the two equations is:

$$\begin{aligned} AB_1 + BC_1 + ABD_1 &= -F_1 \\ AB_2 + BC_2 + ABD_2 &= -F_2 \end{aligned} \qquad \text{(I-11)}$$

where:

$$B_1 = \sum_{k=0}^{n} b_k \omega_n^k T_k + e^{T\omega_n\zeta} \sum_{k=0}^{n} b_k' \omega_n^k X_k$$

$$C_1 = \sum_{k=0}^{n} c_k \omega_n^k T_k + e^{T\omega_n\zeta} \sum_{k=0}^{n} c_k' \omega_n^k X_k$$

$$D_1 = \sum_{k=0}^{n} d_k \omega_n^k T_k + e^{T\omega_n\zeta} \sum_{k=0}^{n} d_k' \omega_n^k X_k$$

$$F_1 = \sum_{k=0}^{n} f_k \omega_n^k T_k + e^{T\omega_n\zeta} \sum_{k=0}^{n} f_k' \omega_n^k X_k$$

$$B_2 = \sum_{k=0}^{n} b_k \omega_n^{\ k} (\zeta T_k + T_{k-1}) + e^{T\omega_n\zeta} \sum_{k=0}^{n} b'_k \omega_n^{\ k} (\zeta X_k + X_{k-1})$$

$$C_2 = \sum_{k=0}^{n} c_k \omega_n^{\ k} (\zeta T_k + T_{k-1}) + e^{T\omega_n\zeta} \sum_{k=0}^{n} c'_k \omega_n^{\ k} (\zeta X_k + X_{k-1})$$

$$D_2 = \sum_{k=0}^{n} d_k \omega_n^{\ k} (\zeta T_k + T_{k-1}) + e^{T\omega_n\zeta} \sum_{k=0}^{n} d'_k \omega_n^{\ k} (\zeta X_k + X_{k-1})$$

$$F_2 = \sum_{k=0}^{n} f_k \omega_n^{\ k} (\zeta T_k + T_{k-1}) + e^{T\omega_n\zeta} \sum_{k=0}^{n} f'_k \omega_n^{\ k} (\zeta X_k + X_{k-1})$$

5. According to Hollister, [2], then:

$$B = \frac{-(\Delta_{FD} + \Delta_{CB}) \pm \sqrt{(\Delta_{FD} + \Delta_{CB})^2 - 4\Delta_{CD}\Delta_{FB}}}{2\Delta_{CD}}$$

and after B is defined:

$$A = \frac{F_1 + C_1 B}{B_1 + D_1 B} = \frac{F_2 + C_2 B}{B_2 + D_2 B}$$

where:

$$\Delta_{CD} = \begin{vmatrix} C_1 & C_2 \\ D_1 & D_2 \end{vmatrix} , \quad \Delta_{FD} = \begin{vmatrix} F_1 & F_2 \\ D_1 & D_2 \end{vmatrix} , \text{ etc.}$$

IV. STABILITY ANALYSIS IN THE PARAMETER PLANE

In this section the stability concepts in the parameter plane are built very qualitatively, at first for a delay-free system. These concepts are based on ideas similar to the ones used in conformal mapping.

For the case of systems with time delay, Karmarkar's method, based on Cauchy's principle of Argument or Michailov's test and its extension by Pontryagin, is followed and used to examine the stability region of the system shown in figure 6-1.

A. SYSTEMS WITH NO TRANSPORT LAG

1. Given a certain polynomial CE(s) in "s", and of nth degree. In general out of the n roots of this polynomial, r_s will lie in the shaded region of figure 4-1a and r_u in the unshaded one, but

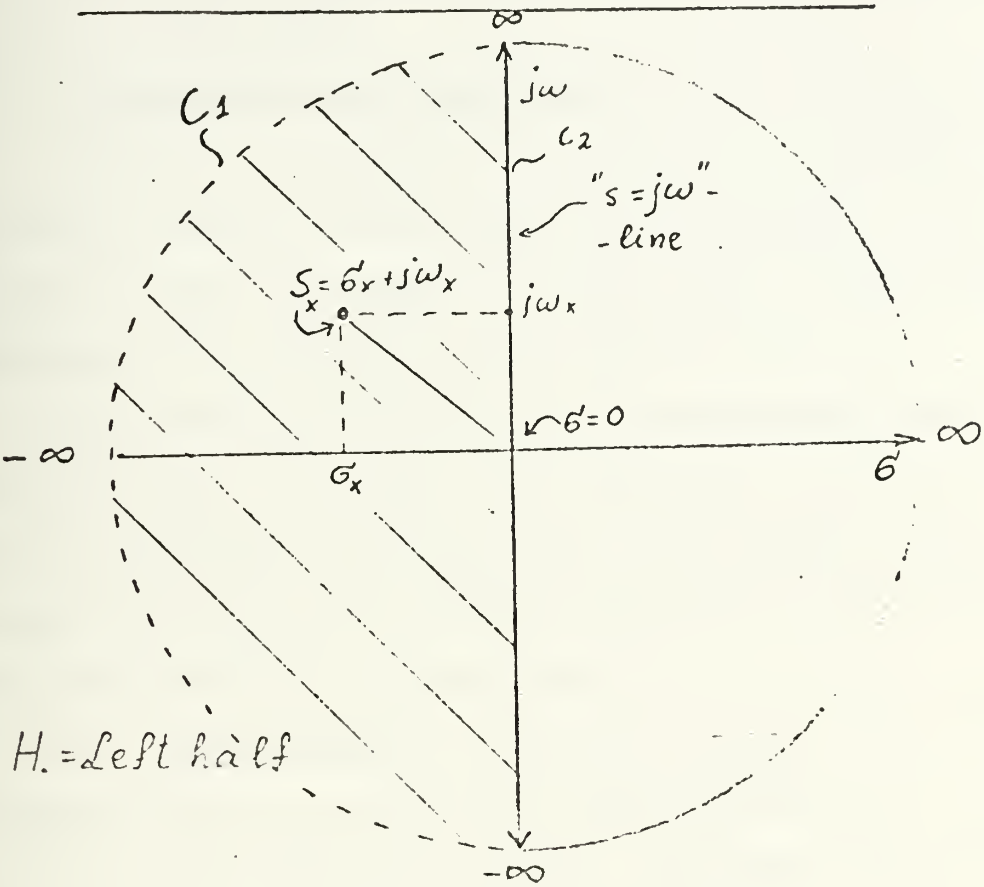
$$n = r_u + r_s \tag{IV-1}$$

So the entire "s" space is divided into two sub-spaces, by the $s = j\omega$ line and the $\sigma = 0$ point, which in a sense are their complex and real root boundaries.

2. Now considering the closed contour c, {consisting of c_1 and c_2 ($\equiv$ the $j\omega$ axis) as shown in figure 4-1a}, we have to realize that it maps in the AB space by appropriate mapping of its boundary. But since the CE(s) polynomial serves as the mapping function, it follows that for different characteristic equations the $\sigma = 0$ and $s = j\omega$ boundaries will map differently. It should also be noted that these boundaries will generally decompose the whole AB space into a finite number δ of regions R_i, each of which will contain i roots in it. So:

$$\sum_{1}^{\delta} i = n \tag{IV-2}$$

L.H.[*1] "S-SPACE" DOMAIN



*1 L.H. = Left half

FIGURE 4-1a

MAPPING IN THE "A,B" PLANE

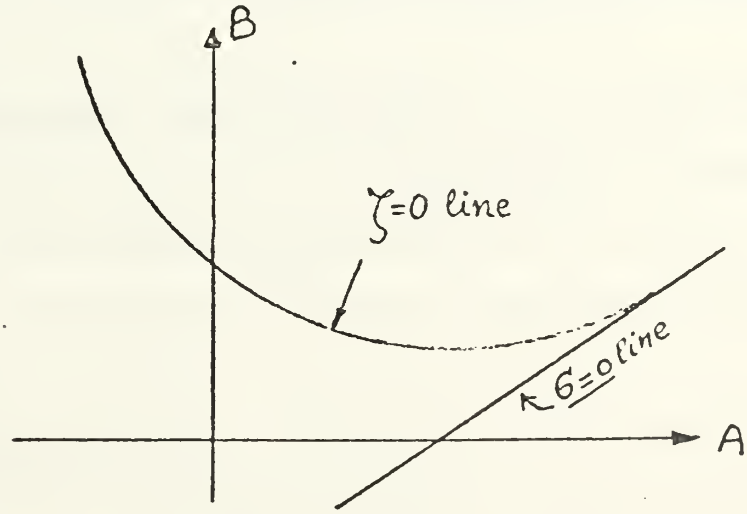


FIGURE 4-1b

3. The whole purpose of the stability analysis is to define the number of roots existing in each of the regions R_i. (In other words, "to determine the region index".)

4. Contour Shading (The Shading Rule).

Consider the mapping process of figure 4-1c, where the complex root boundary is mapped in the AB plane. So if the mapping device lies at point M, it follows that CE(s) has a root $s = j\omega_x$. In other words, "s" maps into "M" via the mapping function CE(s) = 0.

Considering now two points in the M neighborhood, which lie in either side of the $\zeta = 0$ boundary, one realizes that while moving from point M_1 to M_2, at least one complex root in the s plane, moves from s_1 to s_2 or vice versa. It is therefore necessary to orient the boundary regions, according to the following rules:

a. In the s plane, a right handed rectangular system (t,n) is established, where t is the positive tangent direction and n is the normal to it in the appropriate direction. The boundary, then, is shaded always in the direction of positive n. (See figure 4-1c, upper).

Since the complex root boundary in the s-plane is $s = j\omega$, positive t is determined by direction of increase of the variable working along it.

b. In the AB plane, although the (t,n) system is always right-handed in the s-plane, this is not the case for the AB plane. So depending on the particular system under examination the positive n direction can be either side of t. The positive t direction is determined as before. So for:

(1) (t,n) Right-Handed: The positive "n" direction is to the left of positive "t", and the boundary is shaded to the left of positive "t".

ROOT BOUNDARY SHADING IN THE "S AND A,B" PLANES..

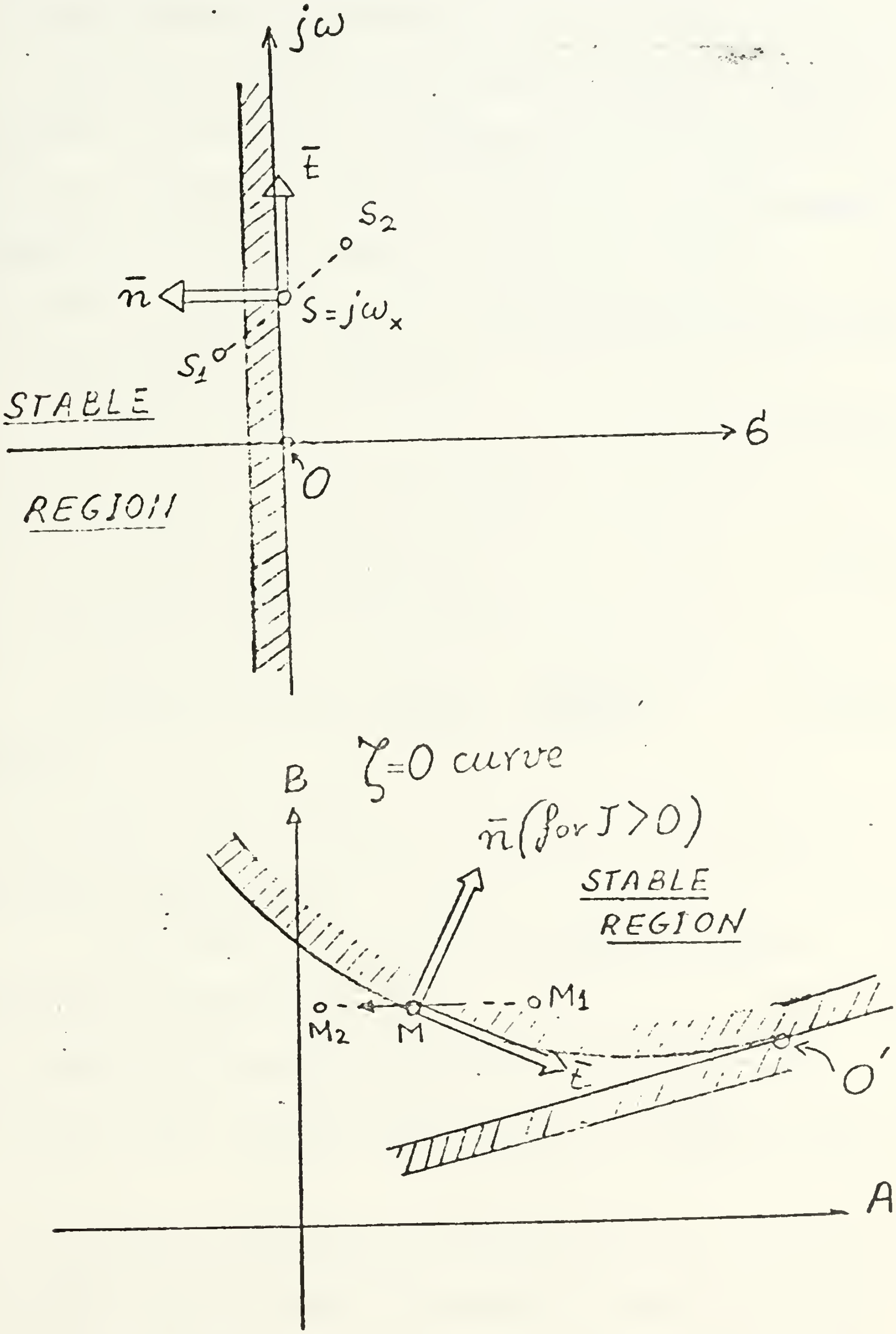


FIGURE 4-1c

(2) <u>(t,n) Left-Handed</u>: Since positive "n" will be directed to the right of positive "t", the boundary is shaded to the right of positive "t".

c. The Significance of Shading.

The shaded portions of the regions in both the s and AB domains correspond, in the specific case examined, to the side of the domain where all the complex roots exist. That is to the stable region since the boundary in "s" was the absolute stability boundary.

d. Rule to Define the (t,n) Co-ordinate System's Orientation

(1) Consider the polynomial

$$F(s) = CE(s) \tag{IV-3a}$$

(2) Substitute $s = \sigma + j\omega$, to get

$$F(\sigma,\omega) = R(\sigma,\omega) + jI(\sigma,\omega) \tag{IV-3b}$$

(3) After forming the Jacobian

$$J = J\left[(R,I)/(A,B)\right] = \left(\frac{\partial R}{\partial A}\right)\left(\frac{\partial I}{\partial B}\right) - \left(\frac{\partial R}{\partial B}\right)\left(\frac{\partial I}{\partial A}\right) \tag{IV-3c}$$

decide as follows:

(a) for $J > 0$, (t,n) is right-handed, therefore shade to the left of positive "t".

(b) for $J < 0$, (t,n) is left-handed, therefore shade to the right of positive "t".

e. Similar rules apply for the real root boundary, (the $\sigma = 0$ curve for this case). However, the following rule applies successfully:

(1) Since at the point of intersection of $s = \sigma$ with the $s = j\omega$ curve ω changes sign, so does the Jacobian J. Therefore, in the AB plane the $\sigma = 0$ curve must change direction of shading at the point of intersection with the $\zeta = 0$ curve.

(2) In the stability region neighborhood of the image O' in the s-domain, the shading of the $\sigma = 0$ and $\zeta = 0$ curves must coincide, since this is the case for the s-domain shading also.

5. The stability region in the AB plane, for a delay-free system, has now been established. For more details one could refer to reference [1].

B. SYSTEMS WITH TIME DELAY

1. In this subsection, Karmarkar's approach, [6], to define if the domain of maximum number of roots is also the stability domain, will be followed since it provides in a single plot the stability domain as well as the test.

2. Karmarkar's stability theorem states the following:

"Consider CE(s) to be a transcendental polynomial:

$$CE(s) = \sum_{p,r}^{m,n} a_{pr} s^p e^{rs} \qquad \text{(IV-4)}$$

where:

a_{pr} are real coefficients

$a_{mn} \neq 0$ (the principal term)

p,n are intergers, and

$$\sum a_{pr} s^p e^{rs} = \begin{bmatrix} s^0 & s^1 \ldots s^n \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m0} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} e^{0s} \\ e^{1s} \\ \vdots \\ e^{ns} \end{bmatrix} \qquad \text{(IV-5)}$$

For CE(s) to be stable it is necessary and sufficient to find an interger K so that, starting with the line $B = b_o$, the total number "λ" of alternating intersection of the $\zeta = 0$ curve with the $A = a_o$ and $B = b_o$ lines

is:

$$\lambda = 4kn + m \tag{IV-6a}$$

when ω ranges over the interval

$$0 \le \omega \le 2\pi k + \epsilon/2, \text{ for } 0 \le \epsilon \le \pi/2,$$

or for $\epsilon=\pi\delta$, $\quad 0 \le \omega \le 2\pi(k + \delta/4)$ for $0 \le \delta \le 1$. (IV-6b)

In figure (4-2), the results of this analysis for the system

$$CE(s) \equiv (s^3 + 2s^2 + s)e^s + as + b = 0$$

is shown. For this particular system one has m = 3 and n = 1. So since "ω" was varied in the range ω = (small ÷ 14) and therefore :

$$\omega = 14 < 2\pi(k + 1/4)_{k=2} = 14.15 \equiv \omega_{max} \text{(where } \omega \text{ in rad/sec)}$$

for k=2, the intersection predicted by the theorem should be

$$\lambda = \Big[(4kn + m) \Big]_{\substack{k=2\\ n=1\\ m=3}} = 11,$$

and this is what is measured, so the system is stable in the domain s bounded by the σ = 0 line, which is the B=0 line as easily one sees from CE(s), and the first encirclement of the ζ=0 curve since the Jacobian (IV-3), is less than zero for this case.

3. The actual derivation of Karmarkar's theorem is based as already mentioned in the following theorems:

a. Mikhailov's Criterion, as a direct application of the Cauchy argument principle, [7].

b. Pontryagin Theorem on the zeros of elementary transcendental functions, [8], and the conditions imposed on A and B when the mapping operating point scribes the $\sum = 0$ curve. (That is equivalent to having roots of the form $s = j\omega$ in the s-plane). For details on his derivation see reference [6].

CONSTANT $\zeta = 0$ AND $\sigma = 0$ CURVES FOR A SYSTEM WITH T. DELAY

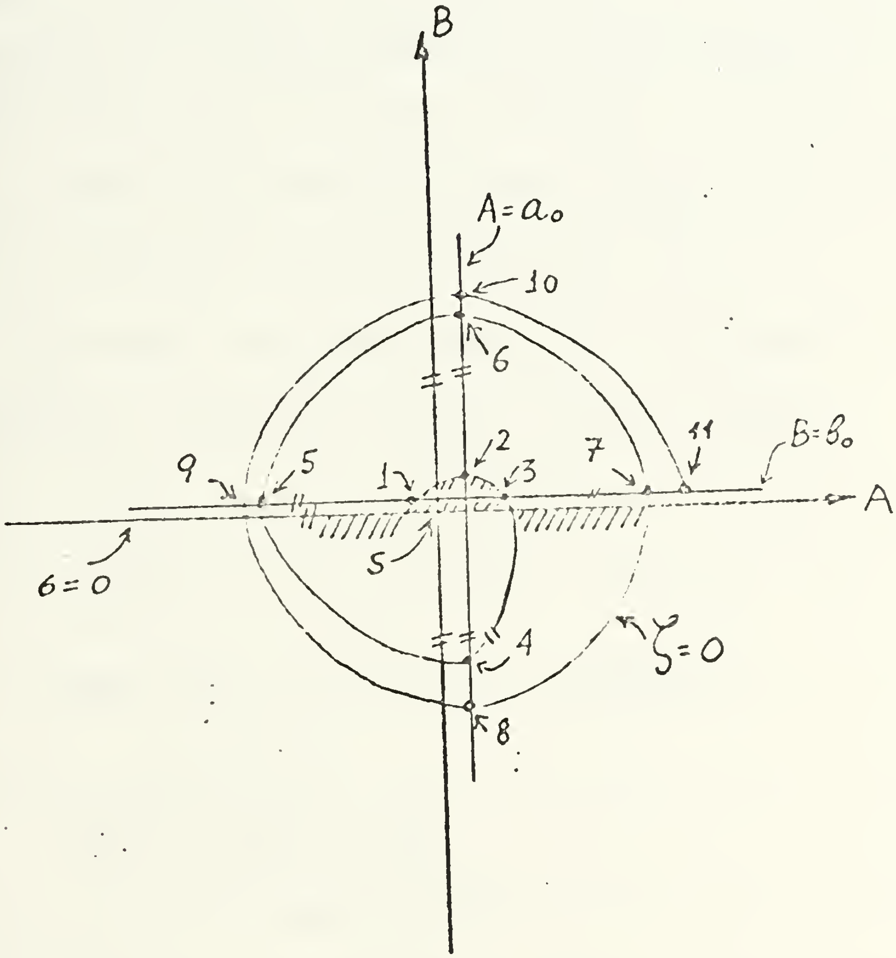


FIGURE 4-2

C. APPLICATION ON AN ACTUAL SYSTEM

1. Consider the stability of the system shown in figure 6-1, for different values of the time delay constant T = 0.0, 0.1, 0.2, 0.3.

It is obvious that for T = 0.0 one can apply methods discussed in section IV A, and for the rest of the time delay constants the methods of section IV B are necessary.

However, a far more practical method of determining the stability of the system is to arbitrarily pick two points in either of the smallest two areas defined by the closed constant $\zeta = 0$[*1] curve and the $\sigma = 0$ straight line, and simulate the system in "t" domain. If for one point the system is stable, the domain it represents is the stable region.

2. The characteristic equation of the system is:

$$CE(s) = \Big[s^6 + 98.4696s^5 + 4941.1680s^4 + 8978.082s^3 + 5086.9961s^2 + 3162.6011s - 147.59\Big] + e^{-Ts}\Big[A(s^3 + .444s^2 + .488s) + B(s^2 + .444s + .488)\Big] \equiv f(s) + e^{-Ts}\Big[A\,g(s) + B\,h(s)\Big] = 0 \qquad \text{(IV-7)}$$

a. In order to obtain constant "σ" curves, let $s = \sigma$.

$$CE(\sigma) = f(\sigma) + e^{-T\sigma}\Big[A\,g(\sigma) + B\,h(\sigma)\Big] = 0$$

and for the $\sigma = 0$ curve one gets:

$$A\,g(0) + B\,h(0) = -f(0)\ ,$$

or $\quad A \times 0 + B \times (.488) = 147.59$

$\therefore$ for finite time delay, the constant "$\sigma = 0$" curve is:

$$B = 302.87 \qquad \text{(IV-8)}$$

---

*1 Notice the equivalence of the $\zeta=0$ and $\Sigma = 0$ curve. In general a "$\Sigma = \alpha$" curve is the loci for roots "$s = \alpha + j\omega$."

b. After multiplying equation (IV-7), by e^{Ts} one gets:

$$e^{Ts} f(s) + e^{0s} \left[A\ g(s) + B\ h(s) \right] = 0 \qquad \text{(IV-7a)}$$

Now since T is not an integer, call $T = t_n/t_d$, where "t_n" and "t_d" are integers,. and substitute into (IV-7a):

$$e^{t_n s'} f(s') + e^{0s'} \left[A\ g(s') + B\ h(s') \right] = 0 \qquad \text{(IV-9a)}$$

where

$$s' = \frac{s}{t_d} \qquad \text{(IV-9b)}$$

As a consequence of equation (IV-5), equation (IV-9a) can also be written as:

$$\begin{aligned}
&(s')^0 \left\{ e^{0s'} \left[0 \times A + .488B \right] + e^{t_n s'} \left[-147.59 \right] \right\} + \\
&(s')^1 \left\{ e^{0s'} \left[.488A + .444B \right] + e^{t_n s'} \left[3162.6011 \right] \right\} + \\
&(s')^2 \left\{ e^{0s'} \left[.444A + B \right] + e^{t_n s'} \left[5086.9961 \right] \right\} + \\
&(s')^3 \left\{ e^{0s'} \left[A + 0 \times B \right] + e^{t_n s'} \left[8978.082 \right] \right\} + \\
&(s')^4 \left\{ e^{0s'} \left[0 \right] + e^{t_n s'} \left[4941.1680 \right] \right\} + \\
&(s')^5 \left\{ e^{0s'} \left[0 \right] + e^{t_n s'} \left[98.4696 \right] \right\} + \\
&(s')^6 \left\{ e^{0s'} \left[0 \right] + e^{t_n s'} \left[1.0 \right] \right\} = 0
\end{aligned} \qquad \text{(IV-9c)}$$

According to the definition of section IV B, the principal term, $(s^m a_{mn} e^{ns})$, is:

$$(s')^6 (1.0) e^{t_n s'} \qquad \text{(IV-10)}$$

where $m = 6$, $n = t_n$ and $a_{mn} = 1.0$.

So for CE(s) to be stable it is needed to find an integer k such that, for $\omega' = (\omega/t_d)$ in the range $0 \le \omega' \le 2\pi k + \frac{\varepsilon}{2} = \pi(2k + \frac{\delta}{4})$,

where:

$$0 \le \epsilon \le \frac{\pi}{2}\text{ , and } 0 \le \delta \le 1\text{ ,}$$

the total number of intersections, λ, of the $\zeta = 0$ curve with the A,B axes moved to a point $P(A=a_o,\ B = b_o)$ is:

$$\lambda = 4kn + m \equiv (4t_n)k + 6$$

In Table 4-1, the columns under the heading "calculation", are "filled in" according to these two formulas giving "λ and ω" for $t_n = (1,2,3)$, which correspond to time delay constant values of T = (.1, .2, .3). The rest of the columns are filled in relation to a computer solution of the "$\zeta=0$" curve vs. frequency and this will be discussed later in this section.

c. Another fact of practical importance to note is that karmarkar's theorem starts counting crossections of the $\zeta=0$ curve with the axes through a testing point in the A,B space, from $\omega=0$. But any digital machine must have an initial value of $\omega\neq0$.

So it is necessary to calculate the origin of the $\zeta=0$ curve for ω tending to zero. This limiting point can be calculated rather easily since for the case of T=0 the time delay effect is not present. But also for $\omega \to 0$ on the $\zeta=0$ curve, the time delay effect is also unimportant since:

$$\lim_{T\to0} e^{-Ts} = \lim_{\omega\to0} e^{-j\omega T} \tag{IV-11}$$

So the origin of the constant $\zeta=0$ curves is the same for a system with a variable time-delay constant, and independent of T.

e. For the particular system of equation (IV-7), one takes the limit by operating on CE(s) as follows:

$$\text{limit of }(\zeta=0,\text{ curve})_{\omega\to0} = \lim_{\omega\to0} CE(j\omega)$$

But this operation will strike out all the "s" dependent terms of CE(s), and therefore

$$\lim_{\omega\to 0} CE(j\omega) = -147.59 + .488B$$

and by equating that to zero the solution is again

$$B = 302.87 \qquad \text{(IV-12)}$$

One should anticipate this continuity of real and complex root boundary since for $\omega\to 0$ the time delay effect is not present and therefore the discussion of section IV A, is valid.

f. It remains now to determine the value of the Jacobian in equation (IV-3c). This Jacobian is in general a function of σ, ω, A, B so for a particular σ

$$\lim_{\sigma\to a} J(\sigma,\omega,A,B) = \lim_{\sigma\to a}\left(\frac{\partial R}{\partial A}\frac{\partial I}{\partial B} - \frac{\partial R}{\partial B}\frac{\partial I}{\partial A}\right) =$$

$$\lim_{\sigma\to a}\frac{\partial R}{\partial A}\cdot\lim_{\sigma\to a}\frac{\partial I}{\partial B} - \lim_{\sigma\to a}\frac{\partial R}{\partial B}\cdot\lim_{\sigma\to a}\frac{\partial I}{\partial A}$$

and because of the nature of the derivative operation, and the well behaved characteristic equation, the order of the limit and derivative operators can be interchanged. So for $\sigma\to a = 0$:

$$J(0,\omega,A,B) = \frac{\partial R(0,\omega)}{\partial A}\cdot\frac{\partial I(0,\omega)}{\partial B} - \frac{\partial R(0,\omega)}{\partial B}\cdot\frac{\partial I(0,\omega)}{\partial A}$$

Now referring to equation (IV-3b), R and I will be computed for T=0, and the same orientation of the (t,n) coordinate system will be assumed for $T \neq 0$ since for $\omega = \omega_0 = 0$ and a differential change in ω, (to cause $\omega = \omega_0 + d\omega = 0 + d\omega$), the (t,n) orientation shouldn't change. But once established the orientation of the ζ=0 curve cannot change. So the orientation for a system with variable time delay is independent of T.

Now taking into account that some terms of R and I will

disappear in the derivative operations, define:

$$R_{mod}(0,\omega) = -.444\omega^2 A - \omega^2 B + .488B$$

$$I_{mod}(0,\omega) = -\omega^3 A + .488\omega A + .444\omega B$$

to be regular R,I polynomials, with the A,B independent terms deleted, and the expression for J is:

$$J(0,\omega) = -(\omega)^5 + .291\omega^3 + .488\omega^2 - .238$$

Considering cases with $\omega > 0$:

$$\lim_{\omega\to\infty} J(0,\omega) = -(\omega)^5 < 0$$

and $$\lim_{\omega\to 0} J(0,\omega) = -.238 < 0$$

So at the "ω" extremes $J < 0$ and due to the nature of the problem the same must occur for intermediate values. Therefore, (t,n), is Left-Handed and for ω increasing the shading goes to the right of the $\zeta=0$ curve.

3. Digital Computer Stability Analysis

a. With all this information available the "$\sigma = 0$" and "$\zeta = 0$" curves are solved with the aid of a digital computer and the program of Appendix D.

b. This was done for T = (0.0, 0.1, 0.2, 0.3), and the results are displayed as follows:

(1) Maximum root regions: In figures from 4-3 to 4-6.

(2) Stability analysis: For a point P in the A,B plane which lies in the general area of the system of section VI is going to to operate, (that is, P = (A = 5722, B = 10568)), the ζ = 0 curve was drawn resulting to crossections with the A_0 = 5722 and B_0 = 10568 lines as shown in figures 4-7,4-8,4-9.

c. By using this information the rest of the table 4-1 was filled in and the partial conclusion was:

(1) For T = 0.1 and 0.2, the system was found to be stable.

(2) For T = 0.3, the "ω range" of the simulation obtained showed that for k=0, k=1 the requirement for stability was not obtained.

4. <u>General Conclusions</u>

The first thing to notice is that for T = 0.0 one gets the biggest maximum root region. This AB region is with no other reservation, the stable region in the AB plane.

As the time delay constant is increased the maximum root region is decreased. But still, this maximum root region is not the stable region. According to Karmarkar's criterion the system with T = 0.3 is not defined stable for k=1 and so one should try k > 1. These results will be checked out and evaluated in section VI, where this practical design problem is encountered in more detail.

5. Notice also that for the one variable parameter, "k", system of figure 2-2 the conclusion was that for increasing time delay constant the maximum k for stability was decreasing. So in a sense the stable "parameter line" was decreasing.

In the two variable parameter system one has also a decreasing area of stability in the parameter plane with increasing T. This should be expected since the physical situation remains the same.

As a result of that we should expect the volume or the hypervolume of stability, if the system can be stable at all for a given time delay constant for a third or higher order variable parameter system, to decrease with "T" increasing.

TABLE 4-1

Delay Constants	t_n	t_d	k selected	Calculation λ	Calculation ω_{max}	Simulation λ	Simulation ω	Remarks
0.1	1	10	3	18	$62.8(3+\frac{\delta}{4})$ = 204.2 rad/sec	18	203.5 rad/sec	Stable
0.2	2	10	1	14	$62.8(1+\frac{\delta}{4})$ = 78.52 rad/sec	14	77.2 rad/sec	Stable
0.3	3	10	1	18	$62.8(1+\frac{\delta}{4})$ = 78.52 rad/sec	19	77.2 rad/sec	Not defined. Should check for k > 1

CONSTANT $\sigma = 0$ AND $\delta = 0$

CURVES FOR $T = 0.0$

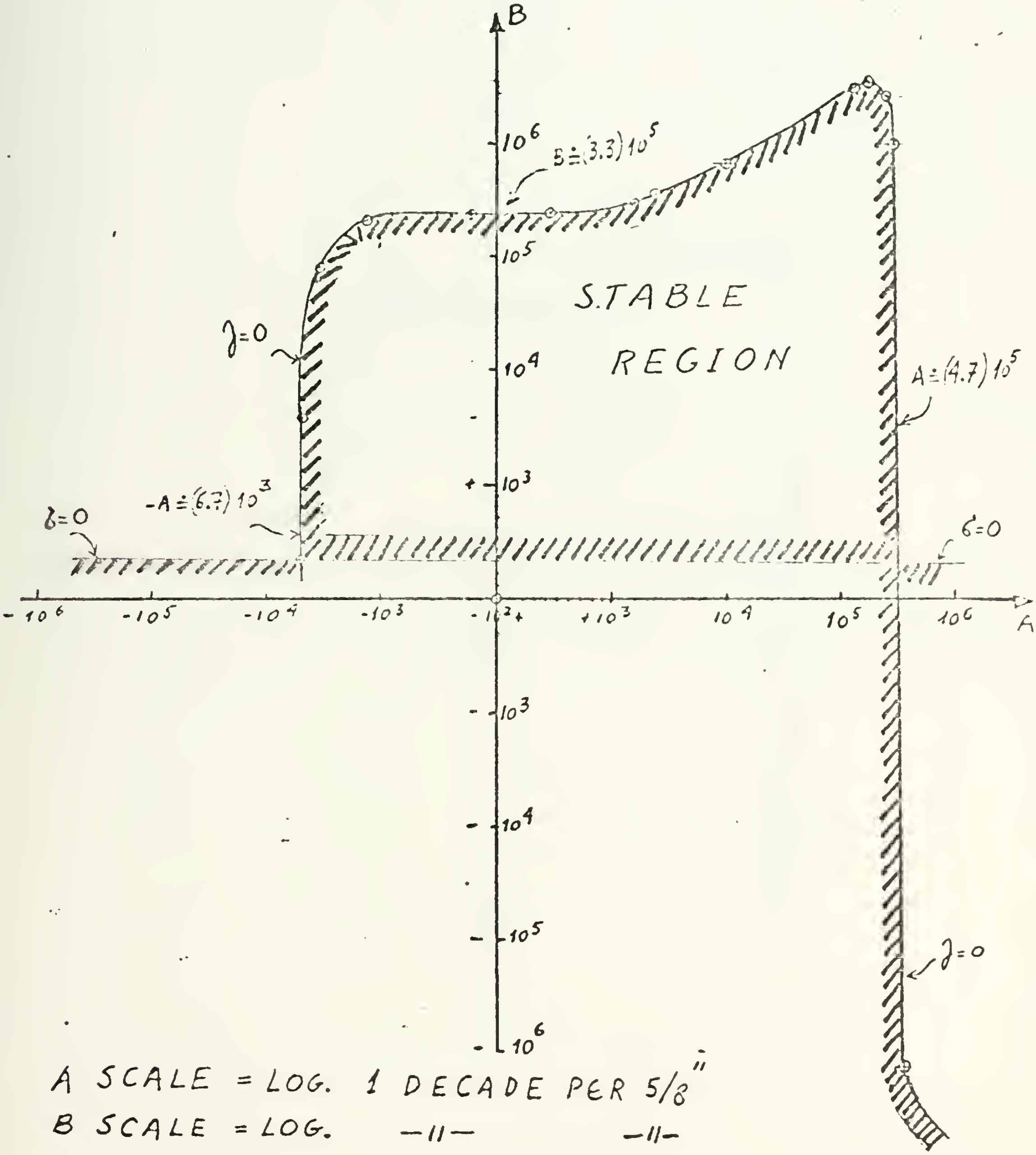
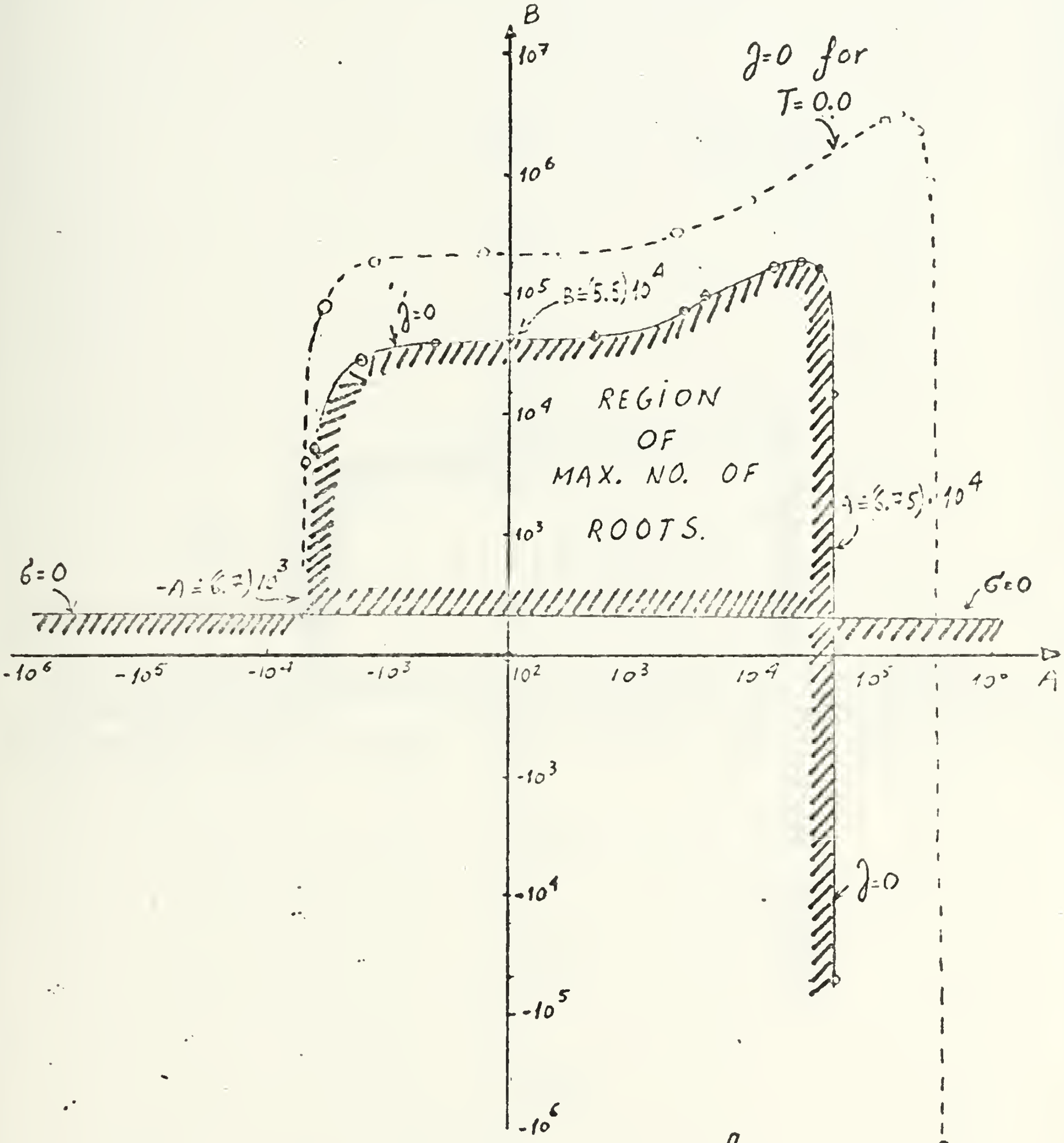


FIGURE 4-3

CONSTANT $\sigma = 0$ AND $\zeta = 0$ CURVES FOR $T = 0.1$

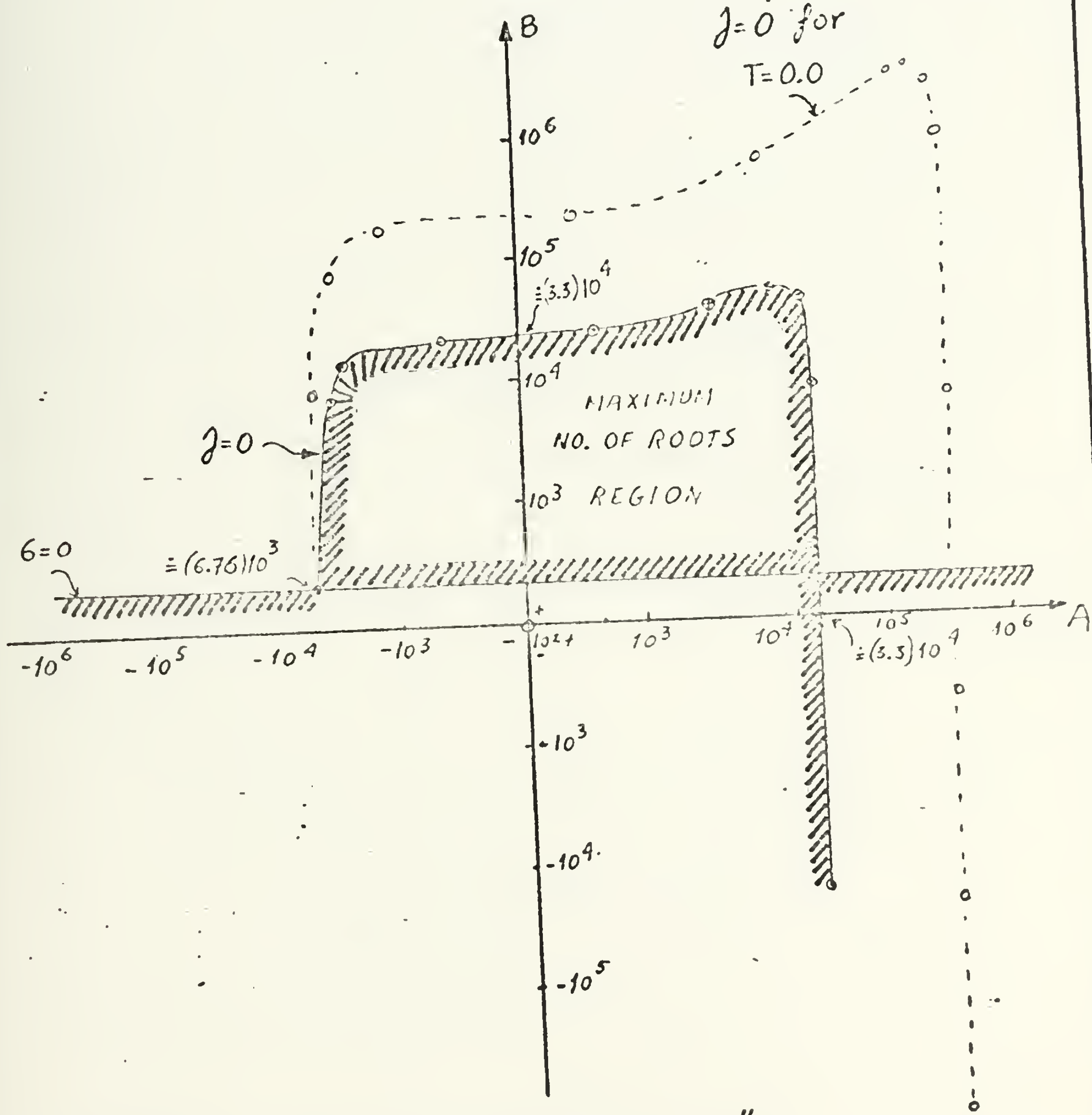


A SCALE = LOG. 1 DECADE PER 5/8"

B SCALE = LOG. —"— —"—

FIGURE 4-4

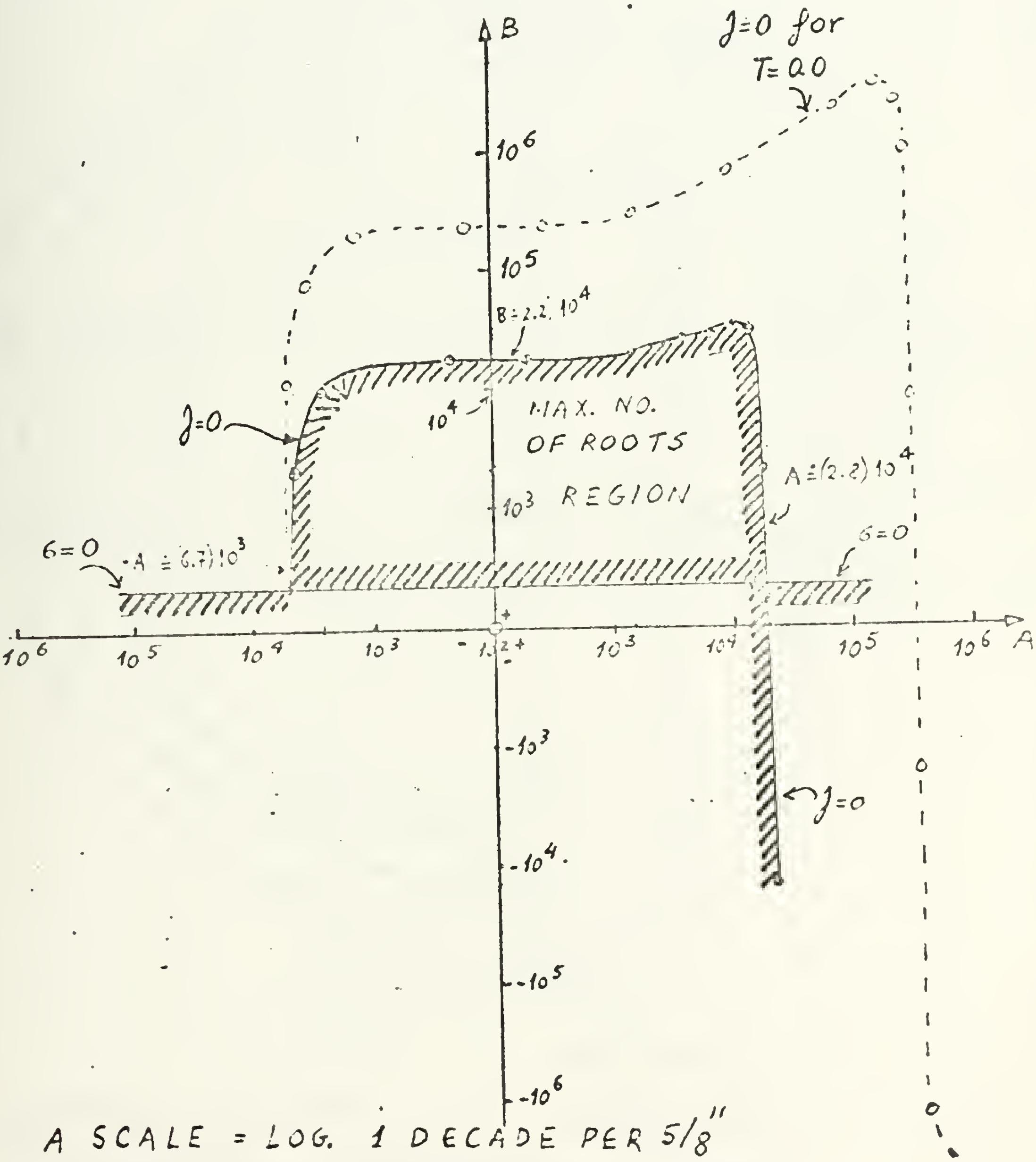
CONSTANT $\sigma = 0$ AND $\mathcal{J} = 0$
CURVES FOR $T = 0.2$



A SCALE = LOG. 1 DECADE PER 5/8"
B SCALE = LOG. —"— —"—

FIGURE 4-5

CONSTANT $\sigma = 0$ AND $\zeta = 0$
CURVES FOR $T = 0.3$



A SCALE = LOG. 1 DECADE PER 5/8"
B SCALE = LOG. —"— —"—

FIGURE 4-6

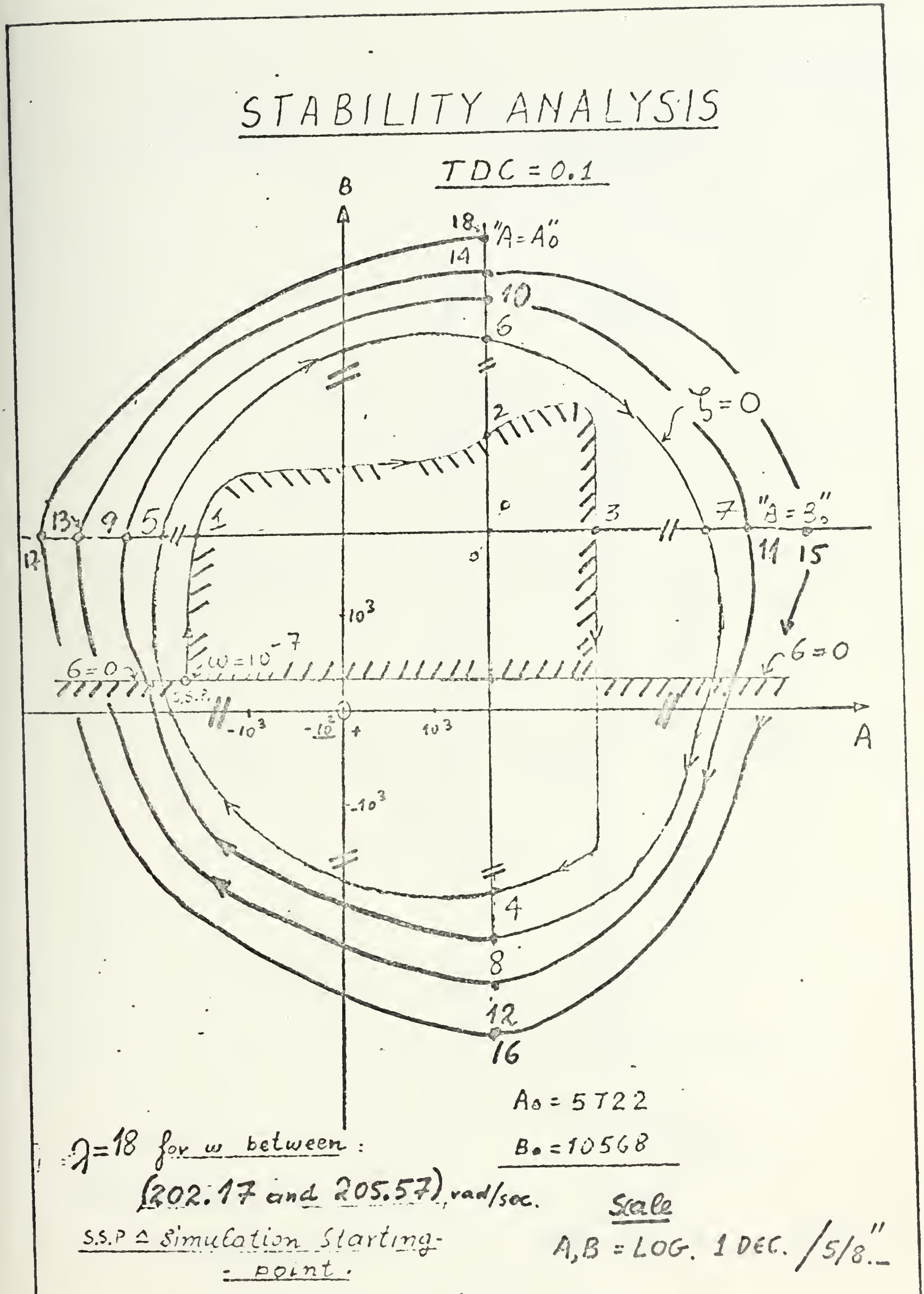
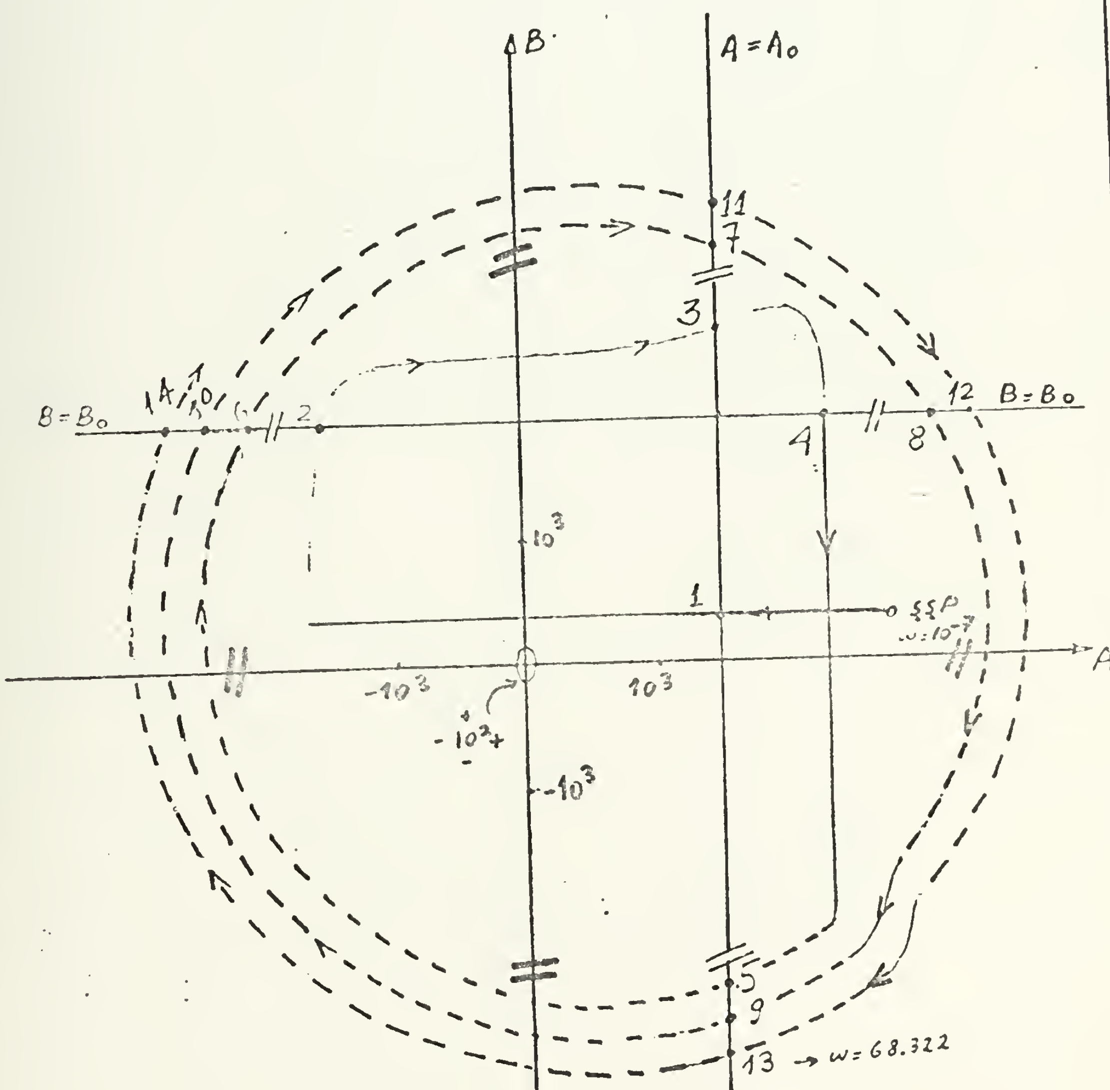

STABILITY ANALYSIS
TDC = 0.1
B
A
"A = A₀"
"B = B₀"
ζ = 0
σ = 0
ω = 10⁻⁷
S.S.P.
10³
-10³
-10²
Ao = 5722
Bo = 10568
η = 18 for ω between : (202.17 and 205.57), rad/sec.
S.S.P ≙ Simulation Starting-point.
Scale
A, B = LOG. 1 DEC. / 5/8"

FIGURE 4-7

STABILITY ANALYSIS

TDC = 0.2



$A_o = 5280$

$B_o = 8279.4$

$\lambda = 14$ for ω between $(75.081 \div 78708)$

S.S.P ≙ Simulation . Starting point

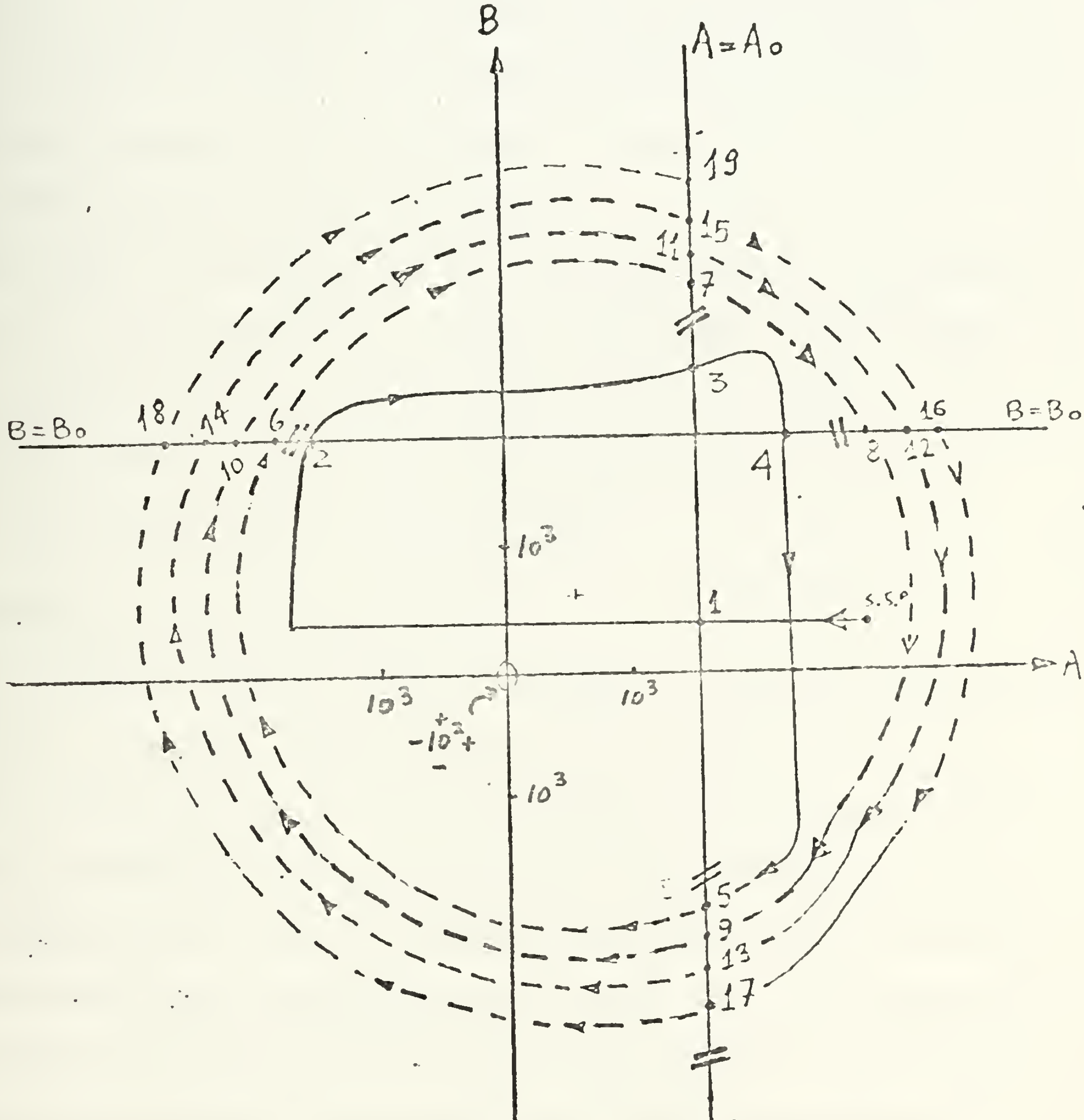
Scale

A,B = LOG. 1 DEC. / 5/8"

FIGURE 4-8

STABILITY ANALYSIS

T.D.C = 0.3



λ = 19 for ω between (75.081 and 73.708)

S.S.P = Simulation Starting point

$A_o = 4463.6$

$B_o = 6064.1$

Scale:

A,B = LOG. 1 DEC. / 5/8"

FIGURE 4-9

V. SYSTEMS WITH THREE VARIABLE PARAMETERS

The purpose of this section is to provide a general way of analysis of systems with three variable parameters to meet specified conditions.

In the analysis to follow, a certain knowledge of the basis of descriptive geometry is needed, so for the reader who is not familiar with the subject, reference [9] is recommended. However, the elementary concepts needed later are presented as well.

The attack to the solution of the most general case of a system with time delay is done indirectly. In fact, a system with characteristic equation linear in A, B, Γ is first analysed so that the concepts of the solution are being built first for this simpler case. The transition then to the basic objective is not difficult.

A. ELEMENTS OF DESCRIPTIVE GEOMETRY

1. Consider a three dimensional orthogonal coordinate system, with axes A, B, Γ. Shown in figure 5-1 is the set of points P_1, P_2, P_3, P_4.

2. The purpose of the descriptive geometry is to provide means of accurate representation and measurements of a three dimensional space in a two dimensional one. So if for this purpose the plane $OB\Gamma$ is selected as the reference where the measurements will take place, the projections of any point of the actual system on the actual $OA\Gamma$ plane, can be shown on the reference without distortion by just a 90° clockwise rotation around the $O\Gamma$ axis. Figure 5-2 then will result in which B and -A as well as A and -B axes coincide. The projections of any point on the $OB\Gamma$ plane are shown by "+" and on the $OA\Gamma$ plane are shown by "⌑".

STRAIGHT LINE IN A THREE DIMENSIONAL SPACE

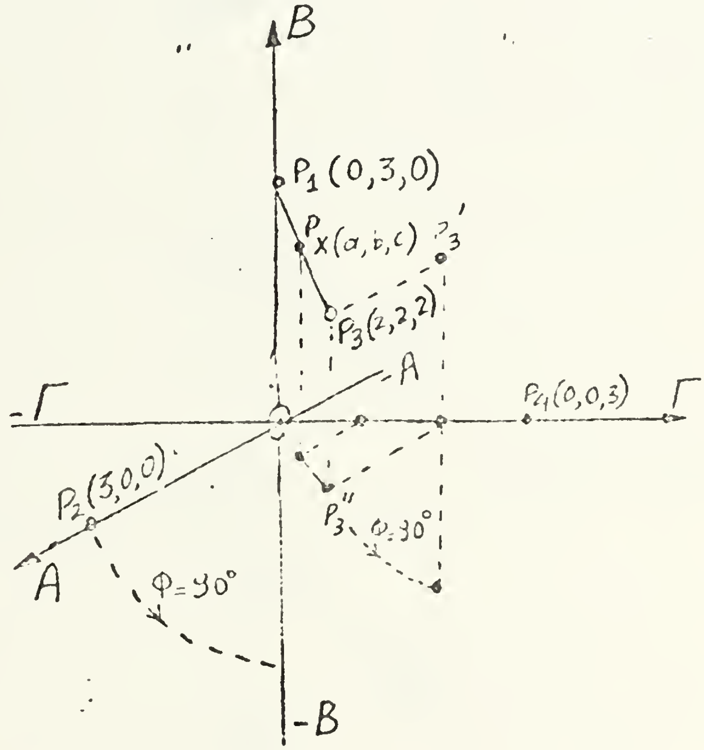


FIGURE 5-1

A 3-DIMENSIONAL STRAIGHT-LINE REPRESENTATION, IN TWO DIMENSIONS

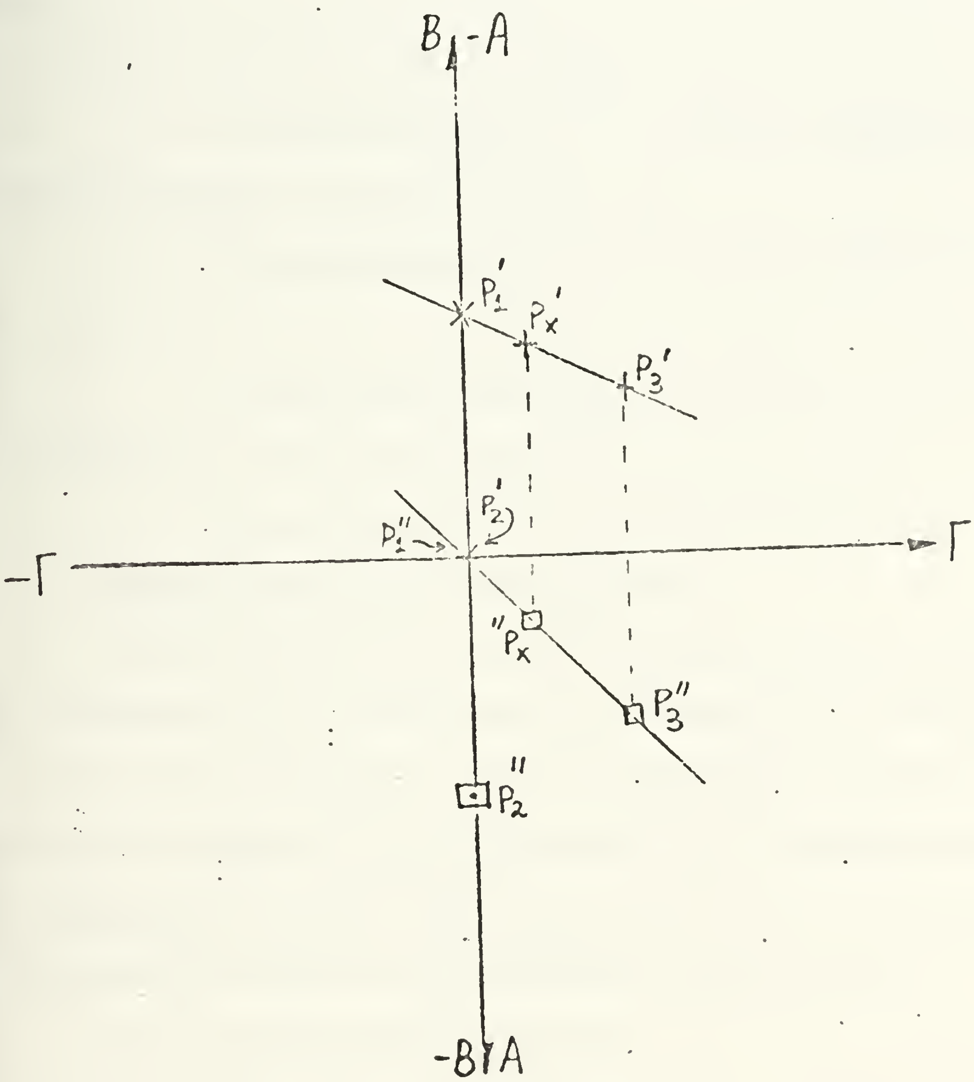


FIGURE 5-2

3. Points on the Reference Plane

By inspection of figures 5-1 and 5-2 one can note the following:

a. A point P in the actual coordinate system, can be represented by two points P' and P" on the reference plane. So we have a one into two forward mapping (and a two to one reverse).

b. These two points (P',P") that form the image of P, lie on a line normal to the OΓ axis, due to the nature of the image generation process.

c. If a three dimensional point lies:

(1) on the OBΓ plane, its P" must lie on the OΓ axis in the descriptive geometry representation (e.g. P_1).

(2) on the OAΓ plane, its P' must lie on the OΓ axis also (e.g. P_2).

(3) on the A=B or A=-B plane, its P' and P" must have the same distance from the OΓ axis (e.g. P_3).

(4) on the OΓ axis, its P' and P" coincide (e.g. P_4).

4. Straight Lines on the Reference Plane

If now points P_1 P_3 are joined together to form the straight line they define in the actual space, it is reasonable to anticipate that the images of this line in the reference plane must be the two straight lines that are defined by (P_1' P_3') and (P_1" P_3"). Furthermore for each point $Px_{(a,b,c)}$ on the three dimensional space there is one and only one set of points Px' and Px" in the reference plane such that:

a. Px' lies on the P_1'P_3' line

b. Px" lies on the P_1"P_3" line

c. If g is the intersection of OG and P' Px" on the reference plane;

$$|g\ Px'| = b\ ,\quad |g\ Px''| = a\ ,\quad |Og| = c\ .$$

5. Planes on the Reference Plane

A plane is represented by two of its lines or three of its points in the 3-D space. So it needs four straight lines or six points in order to be defined in the reference plane. Notice the exceptions when the actual plane is:

a. Normal to the $O\Gamma$ axis. In this case all the four images of any two lines selected to represent it, will coincide and will be normal to the $O\Gamma$ axis.

b. Represented by its intercept lines on the $OB\Gamma$ and $OA\Gamma$ planes. In this case two straight line images will suffice and the other two will coincide with the $O\Gamma$ axis.

6. Lines Other Than Straight, on the Reference Plane

Much of the power of the descriptive geometry representation is lost when one deviates from straight lines or planes. For example, the intersection of a straight line and a plane can be determined by a standard procedure, but of a curved line and a plane the intersection can be determined by trial and error only.

However, any continuous line in actual space will give continuous images in the reference plane and vice versa.

So if one knows that the images (P_1', P_1''), (P_2', P_2''), (P_3', P_3'') and (P_4',P_4'') as shown in figure 5-3b projected by points P_1, P_2, P_3, and P_4 which formed a continuous line in space, then one can anticipate that the projections of the actual line segment would more or less look as shown in figure 5-3b and the approximation of the actual curve shown in figure 5-3a would be more accurate if more points at closer distances were available.

A CURVED LINE IN SPACE

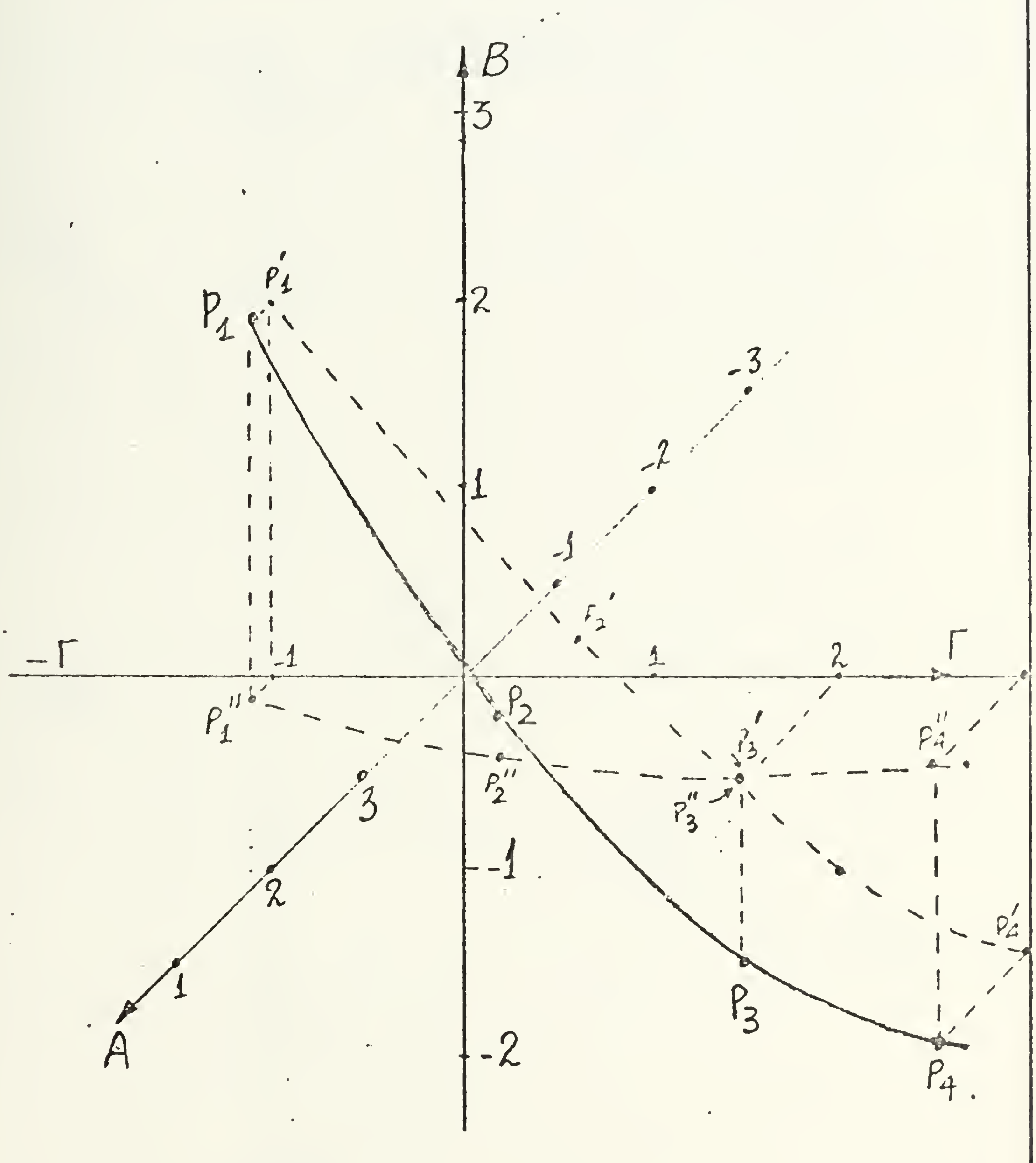


FIGURE 5-3a

A "3-DIM." CURVED LINE REPRESENTATION, IN A "2-DIM" SPACE

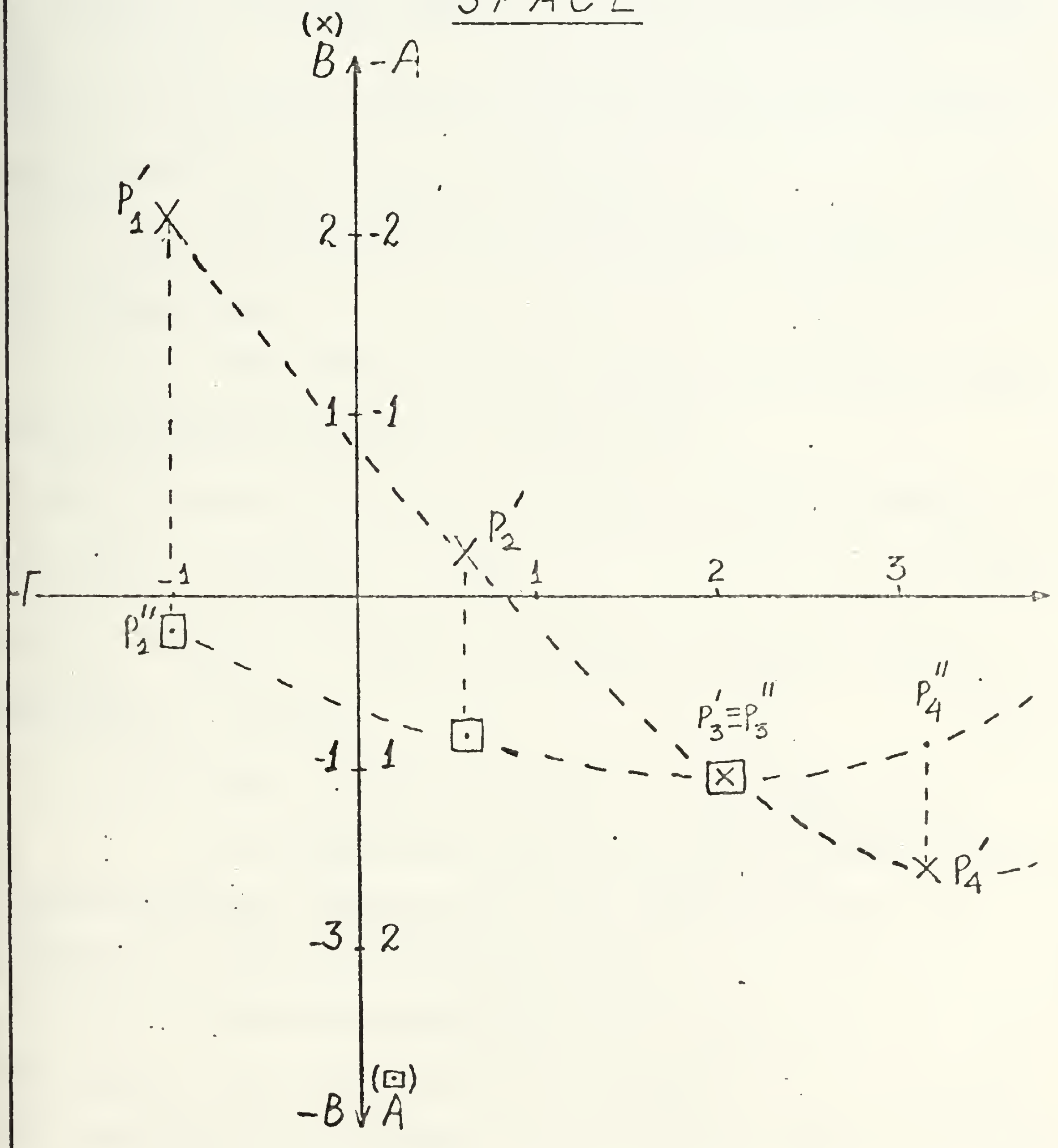


FIGURE 5-36

B. GENERAL THREE VARIABLE PARAMETER SYSTEM WITH TIME DELAY--A CONCEPT

1. A simple extension of the case of a system with two parameters, entails that for the case of three variable parameters A (Alpha), B (Beta), Γ (Gamma), the general characteristic equation would look like:

$$A\left[C_1(s) + e^{-Ts}C_2(s)\right] + B\left[D_1(s) + e^{-Ts}D_2(s)\right] + \Gamma\left[E_1(s) + e^{-Ts}E(s)\right] +$$

$$AB\left[F_1(s) + e^{-Ts}F_2(s)\right] + A\Gamma\left[H_1(s) + e^{-Ts}H_2(s)\right] + B\Gamma\left[I_1(s) + e^{-Ts}I_2(s)\right] +$$

$$AB\Gamma\left[K_1(s) + e^{-Ts}K_2(s)\right] + \left[L_1(s) + e^{-Ts}L_2(s)\right] = 0 \qquad \text{(V-1)}$$

2. The solution for the ζ and ω_n curves for this system must follow the following steps:

a. First derive two sets of equations (separating once again reals and imaginaries) in terms of ζ and ω_n, e.g.

$$A\alpha_1(\zeta,\omega_n) + B\delta_1(\zeta,\omega_n) + \Gamma\epsilon_1(\zeta,\omega_n) + AB\phi_1(\zeta,\omega_n) + A\Gamma\eta_1(\zeta,\omega_n) + B\Gamma\varphi_1(\zeta,\omega_n) +$$

$$AB\Gamma\mu_1(\zeta,\omega_n) = -\xi_1(\zeta,\omega_n) \quad , \ \underline{\text{and}} \qquad \text{(V-2a)}$$

$$A\alpha_2(\zeta,\omega_n) + B\delta_2(\zeta,\omega_n) + \Gamma\epsilon_2(\zeta,\omega_n) + AB\phi_2(\zeta,\omega_n) + A\Gamma\eta_2(\zeta,\omega_n) + B\Gamma\varphi_1(\zeta,\omega_n) +$$

$$AB\Gamma\mu_2(\zeta,\omega_n) = -\xi_2(\zeta,\omega_n) \qquad \text{(V-2b)}$$

b. Follow the same steps as in the two parameter case by:

(1) Giving actual values to ω_n and ζ to form the certain "constant performance characteristic curve," therefore making $\alpha_1, \delta_1, \ldots$ and α_2, $\delta_2, \ldots$ constant for each "ζ,ω_n" pair.

(2) Solving the two equations of the form of (V-2) but with constant coefficients, for the three unknowns A, B, Γ, by applying some restriction on the solution, e.g. by watching a certain plane cut of the A, B, Γ space.

c. Put the partial plots together to reconstruct if possible the whole picture in the A, B, Γ space.

3. This idea of taking cuts on the three parameter space is not new and in fact solutions by quantizing one of the three parameters (which is equivalent to subsequent cuts normal to one of the three parameter axes), have been successfully presented. The only difference is that this character of analysis has a trial and error characteristic which of course is reduced if the physical limitations of one of the variables are more or less narrow.

4. However, by taking enough cuts with planes passing through one of the coordinate axes of the parameter space, one can solve an equivalent number of two dimensional problems which can be displayed on the same reference plane by use of simple descriptive geometry ideas.

5. These ideas can be very easily grasped for the case of a system simpler than the general.

C. A SYSTEM WITH LINEAR CHARACTERISTIC EQUATION IN A, B, AND Γ

1. Consider, now, equations (V-2) for the special case that AB, $A\Gamma$, $B\Gamma$, $AB\Gamma$ do not exist. In this case:

$$A\alpha_1(\zeta,\omega_n) + B\delta_1(\zeta,\omega_n) + \Gamma\varepsilon_1(\zeta,\omega_n) = -\xi_1(\zeta,\omega_n)$$
$$A\alpha_2(\zeta,\omega_n) + B\delta_2(\zeta,\omega_n) + \Gamma\varepsilon_2(\zeta,\omega_n) = -\xi_2(\zeta,\omega_n) \qquad \text{(V-2a)}$$

2. Consider, now, that one wants to trace the "constant ω_n loci" for $\omega_n = \omega_{n_1}$ with either $\zeta = \zeta_1 \div \zeta_n$ or $\zeta = \zeta_1, \zeta_2, \ldots, \zeta_n$ for a continuous or quantized solution (this makes no difference). That is, the following set of equations must be solved, "n" times over the "ζ" range desired:

$$\begin{bmatrix} A\alpha_1(\zeta,\omega_n) + B\beta_1(\zeta,\omega_n) + \Gamma\varepsilon_1(\zeta,\omega_n) = -\xi_1(\zeta,\omega_n) \\ A\alpha_2(\zeta,\omega_n) + B\beta_1(\zeta,\omega_n) + \Gamma\varepsilon_1(\zeta,\omega_n) = -\xi_2(\zeta,\omega_n) \end{bmatrix}_{\zeta = \zeta_1}^{\zeta = \zeta_n} \qquad \text{(V-2b)}$$

So one must solve "n" times for the intersections of two planes in the

ABΓ space. In other words the constant $\omega_n = \omega_{n1}$ loci is a set of "n" straight lines, with n going to ∞ if ζ is varied continuously in the range $(\zeta_1 \div \zeta_n)$.

3. But by letting B = 0, for any arbitrary $\zeta = \zeta_{ar}$ from the above range, in both of equations (V-2b) and solving for:

$$A\alpha_1(\zeta_{ar},\omega_{n1}) + \Gamma\epsilon_1(\zeta_{ar},\omega_{n1}) = -\xi_1(\zeta_{ar},\omega_{n1})$$

$$A\alpha_2(\zeta_{ar},\omega_{n1}) + \Gamma\epsilon_1(\zeta_{ar},\omega_{n1}) = -\xi_2(\zeta_{ar},\omega_{n1})$$

one gets a point P_1 in the OAΓ plane which belongs also to the desired solutions of V-2b for $\zeta = \zeta_{arbitrary}$ (see figure 5-4a).

Analogously for A = 0, one can get a point P_2, on the OBΓ plane, which point will also belong to the desired straight line solution of (V-2b) which after all is completely defined.

4. So this solution can now be projected in the descriptive geometry reference plane (Figure 5-4b), and mapped by its two straight line images. These two images define any point that happens to have a pair of performance characteristics $\omega_n = \omega_{n1}$ and $\zeta = \zeta_{arb}$, $\left[\zeta_{arb} \in (\zeta_1 \div \zeta_n)\right]$, for example a point Px in figure 5-4b with A = 1, B = 1, Γ = 4 would meet those requirements. Some work on this has been presented by Thaler and Cadena.

5. The fortunate fact to note though, is that one can keep track and interpret the meaning of the two points P_1, P_2 as part of a simple infinite solution for $\omega_n = \omega_{n1}$ and $\zeta = \zeta_{arb}$, by simply locating the points P_2' , and P_1'' on the reference plane, since P_2'' and P_1' can be located by drawing the normals to OΓ for each of them. One should of course indicate which pair of ω_{n1} and ζ_{arb} was used for this set of points.

6. But now looking back again in the whole process one recognizes

STRAGHT LINE IN A "3-DIM" SPACE

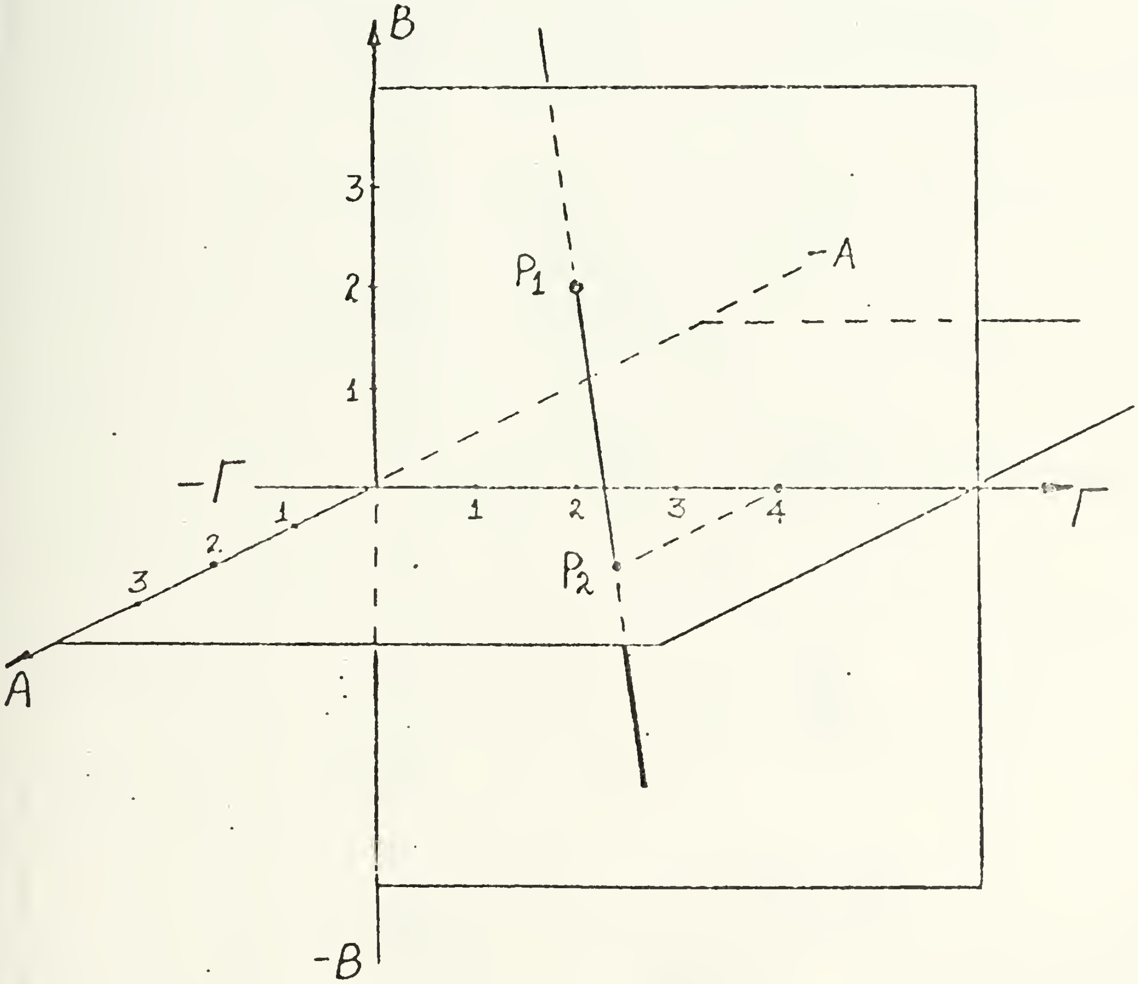


FIGURE 5-4a.

PARAMETER READING FROM LINE PROJECTIONS

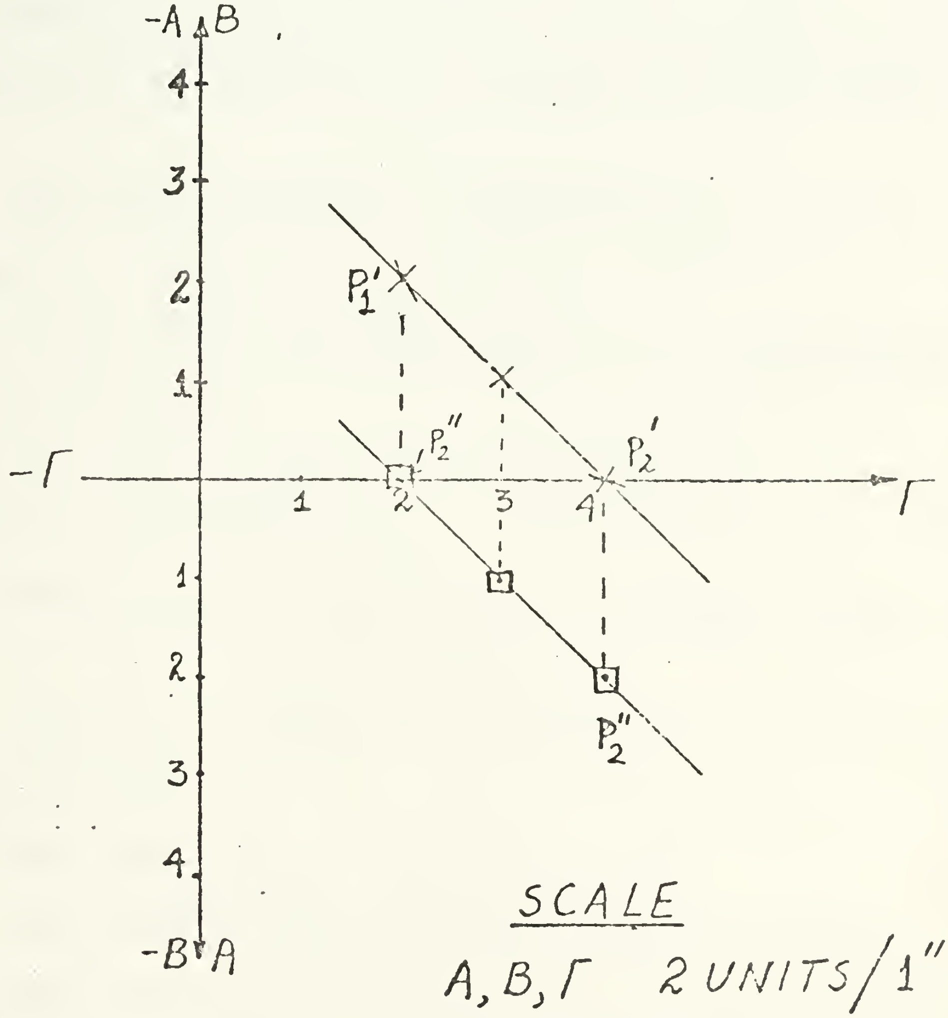


FIGURE 5-4b

that what is necessary to solve this problem is to solve simultaneously for each set of values of $\omega_n = \omega_{n1}$ and $\zeta = \zeta_{arbitrary}$, a two dimensional, linear parameter plane problem in A,Γ and B,Γ and to locate also simultaneously the two corresponding points on the same planar plot. This process will continue regularly in order to complete two individual planar grids of constant ω_n and constant ζ curves as shown in Figure 5-5.

7. It is important to plot each of the two grids with different locus indicators, eg for the OBΓ and for the OAΓ projections.

8. If, after the two parameter plane grids are plotted, one wants information about the $\omega_n = \omega_{n1}$, $\zeta = \zeta_1$, A,B,Γ space locus, the procedure is:

a. First have in mind that the plot obtained lies in the reference plane.

b. Second, locate the two points for each grid, corresponding to the $\omega_n = \omega_{n1}$, $\zeta = \zeta_1$ loci.

c. Draw the two normals on OΓ to locate the associated second projection for each image.

d. Draw the lines ℓ' and ℓ'' which are the images of the desired locus.

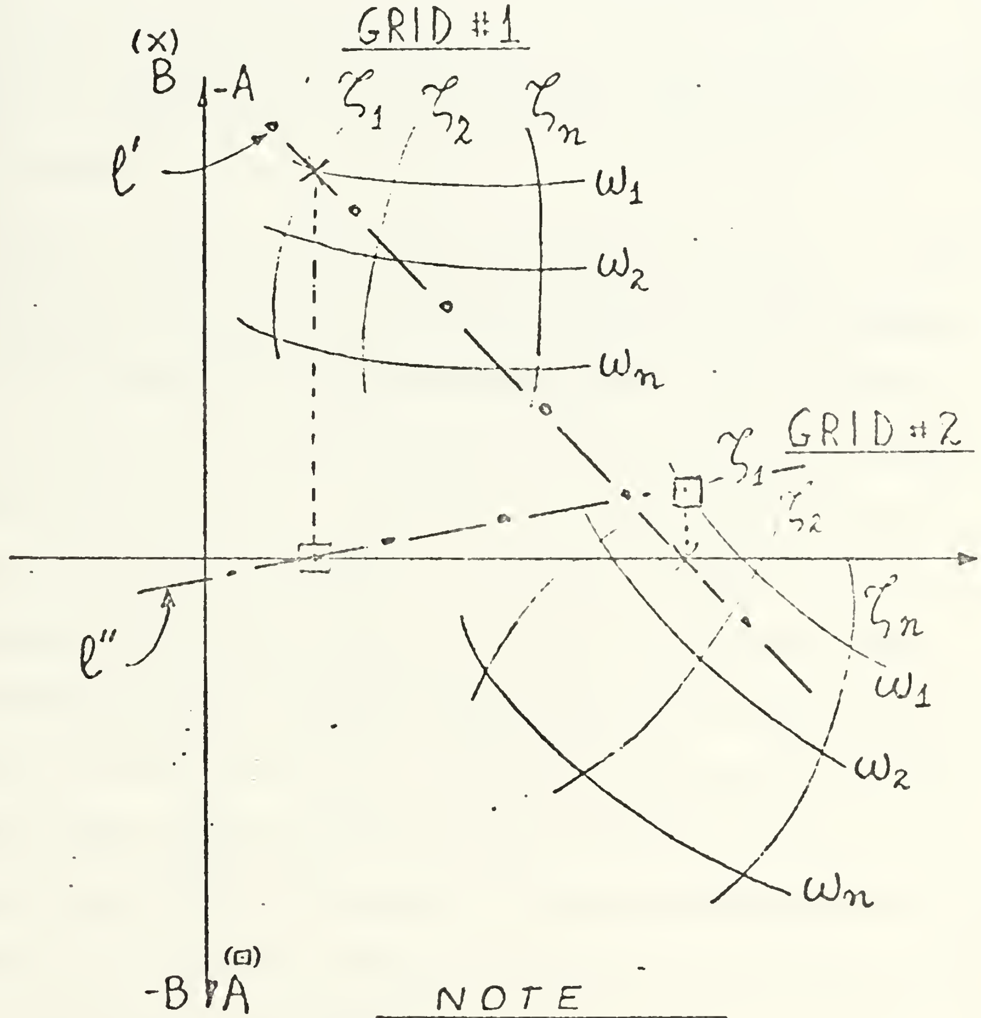
e. In order to read one of the infinite sets of A,B,Γ values that give $\omega_n = \omega_{n1}$ and $\zeta = \zeta_1$:

(1) First, select a Γ value, say Γ = Og

(2) To get the other two parameter's values, sketch a line normal to gOΓ at the point g. Then: A = ga, B = gb, the intercepts of the line bga on the projections ℓ'' and ℓ', respectively.

9. An example of an actual simple linear-characteristic equation system follows.

SUPERPOSITION OF GRIDS COMPOSED OF CONSTANT "ω" AND "ζ" CURVES



NOTE

GRID #1 came from a BOΓ projection so it contains BETA's at ζ, ω intersections.-

GRID #2 Equivalently contains ALPHA's at ζ, ω intersections.

FIGURE 5-5

D. EXAMPLE PROBLEM

1. The closed loop transfer function for the system in figure 5-6 is:

$$\frac{C}{R} = \frac{\Gamma'(s-1)}{(s^2+3s+2)+\Gamma'(s-1)(A's+B')} = \frac{(s-1)}{\Gamma(s^2+3s+2)+(s-1)(A's+B+1)}$$

where

$$\Gamma = \frac{1}{\Gamma'} \quad , \text{ and } \quad B = B' - 1.$$

2. So the characteristic equation for this system being of the form previously discussed is:

$$A'(s^2 - s) + B(s - 1) + \Gamma(s^2 + 3s + 2) + (s - 1) = 0 \qquad \text{(V-3)}$$

3. Practical Considerations on Applying Parameter Plane Methods

At this stage it is of importance to notice that a digital computer program for solving and drawing the constant ω_n and constant ζ loci is absolutely necessary. One could use either a program based on the equations of paragraph E of this section (V), or could use twice a program designed to solve and draw for the case of two variable parameters, and then superimpose the two drawings (one for an $A0\Gamma$-plane cut and the other for a $B0\Gamma$-plane cut). But even if one uses the two variable parameter plane program, difficulties may arise in selecting the proper range of A,B,Γ (that is the scale of the graph) basically for the desired range of "ω" and "ζ". However, even if one or two curves are drawn, the task then is even simpler.

To get a first rough estimate of the appropriate range[*2] (under

---

[*2] Refer to the conclusions (section V-G) for comments regarding root loci methods versus parameter space approach, and also note that the solutions by the root loci techniques in subparagraphs "a" and "b" to follow, are just particular arbitrary solutions and are not likely to be the desired solution.

BLOCK DIAGRAM FOR A "3-VAR. PARAMETER", DELAY-FREE SYSTEM

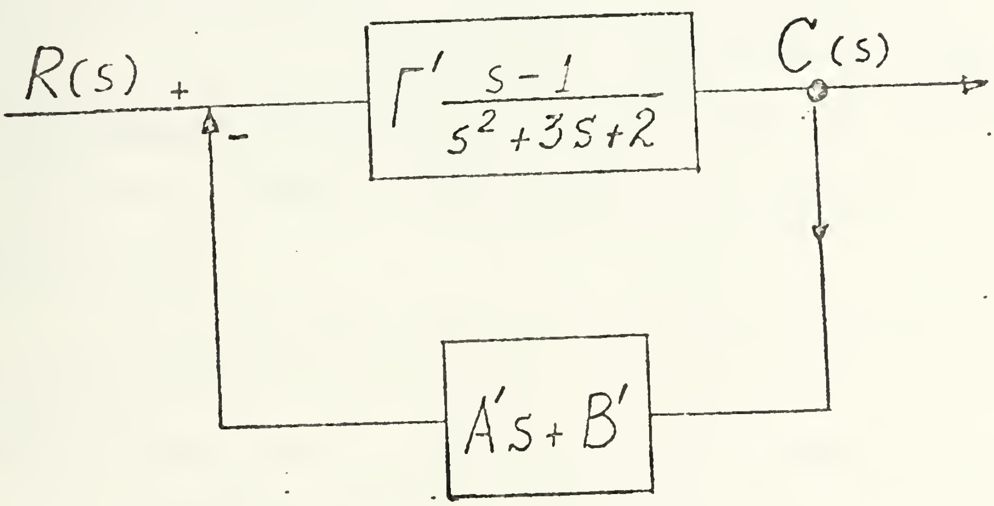


FIGURE 5-6

the assumption that some test runs with the computer program of Appendix D will follow), one can proceed in many ways. Three of them are as follows:

a. The Case of Forward Loop or Feed-Back Compensation with the Forward Gain as One of the Variable Parameters

(1) Draw the open loop root locus with <u>no</u> compensation unit included

(2) Acknowledging the fact that except for the forward gain, the rest of the variable parameters, once fixed, define possible compensation poles and zeros, try to find a combination of these parameters that would change the open loop root locus to be stable for the desired working frequency with a ζ approximately equal to 0.707.

(3) As a very quick check on the approximate forward gain range apply the Routh-Hurwitz stability criterion for the closed loop characteristic equation. Sometimes this is not enough and one should draw the root locus.

(4) The rough scale to be used for the first time should be as a rule of thumb[*3] ten to a hundred times the values of the compensation variables selected and the maximum stable forward gain.

b. Compensation (Forward Loop or Feed-Back) with a Specified Forward Gain

(1) Insert a variable gain parameter in place of the specified forward gain value.

(2) Repeat the procedure for the previous case, but accepting the resulting scale if the specified gain is as a rule of thumb

---

[*3] The rule of thumb proposed, might of course not work for $\frac{d\zeta}{dk}$ very large or very small, and therefore some change should then be necessary depending of the particular system.

twenty percent less than the maximum stable gain obtained. Otherwise, try a new combination of the compensation variables. The reason of this procedure is to assure an A,B pair which would give under the operating "k", at least one complex pole pair in the region of ζ = (.3 ÷ .8), because the possibility of not getting a constant ζ curve is reduced when this pair is forced in the corresponding s-domain area, for "ζ" around 0.707.

c. The Closed Loop Characteristic Equation Technique

When only the closed loop characteristic equation is given, and one has no idea about the actual system, the following procedure could be followed:

(1) Obtain three Root-Locus Plots, each for one of the three parameters, as the only variable and the remaining two set equal to a convenient constant (say unity).

(2) For a certain range of "ζ" and "ω_n" desired, define the corresponding range of each of the variables in each of the root-locus plats. This will give an indication of the scaling of the problem to start with.

One should also have in mind, once engaged with a time delay problem, to set the time decay constant equal to zero, to apply the method discussed above.

4. Practical Considerations on Applying Parameter Methods of Paragraph C

Considering equation (V-3), and having in mind the discussion of paragraph C, one comes to the conclusion that if a BOΓ cut of the parameter space is tried for, the particular system of figure 5-6 (i.e. A=0) is physically restricted from having any ζ, since the roots of the GH function will all lie on the σ axis in an "s-plane" representation.

So there is a physical limitation in applying the technique, which one can overcome by a change of variable $A' = A + 1$, thus maintaining the tachometer feedback of the system for the case of $A = 0$. In this case, equation (V-3) becomes:

$$A(s^2 - s) + B(s - 1) + \Gamma(s^2 + 3s + 2) + (s^2 - 1) = 0 \quad \text{(V-4)}$$

5. Having in mind all the practical considerations above, one can proceed to specify an approximate set of A, B, Γ range for a first try of the parameter space solution.

a. From figure 5-7 (following the procedure section V-D3), one can see a good value for the compensation zero to be at $\sigma = 2$, (or $\sigma = 3$, etc.). This can result for $A' = 2$, $B' = -4$ and equivalently for $A = 1$, $B = -5$.

b. Since the Routh Hurwitz criterion is going to give an expression of $\Gamma'_{max} < M_{gain}$, for $M_{gain} > 0$, it follows that:

$$\Gamma_{max} = \left(\frac{1}{\Gamma'_{max}}\right) > M_{gain}$$

for stability, and this is not enough. (Root-locus is needed).

c. For $A = 1$, $B = -5$, the second order system of equation (V-4), becomes:

$$s^2 + \frac{3(\Gamma - 2)s}{\Gamma + 2} + 2$$

It follows then that for a $\zeta = 1/\sqrt{2}$ and $\omega_n = \sqrt{2}$ a $\Gamma = 10$ is needed.

d. So as a first try the following ranges should be tried out:

(1) $|A| \leq 10$

(2) $|B| \leq 50$

(3) $|\Gamma| \leq 20$

ROOT-LOCUS SKETCH

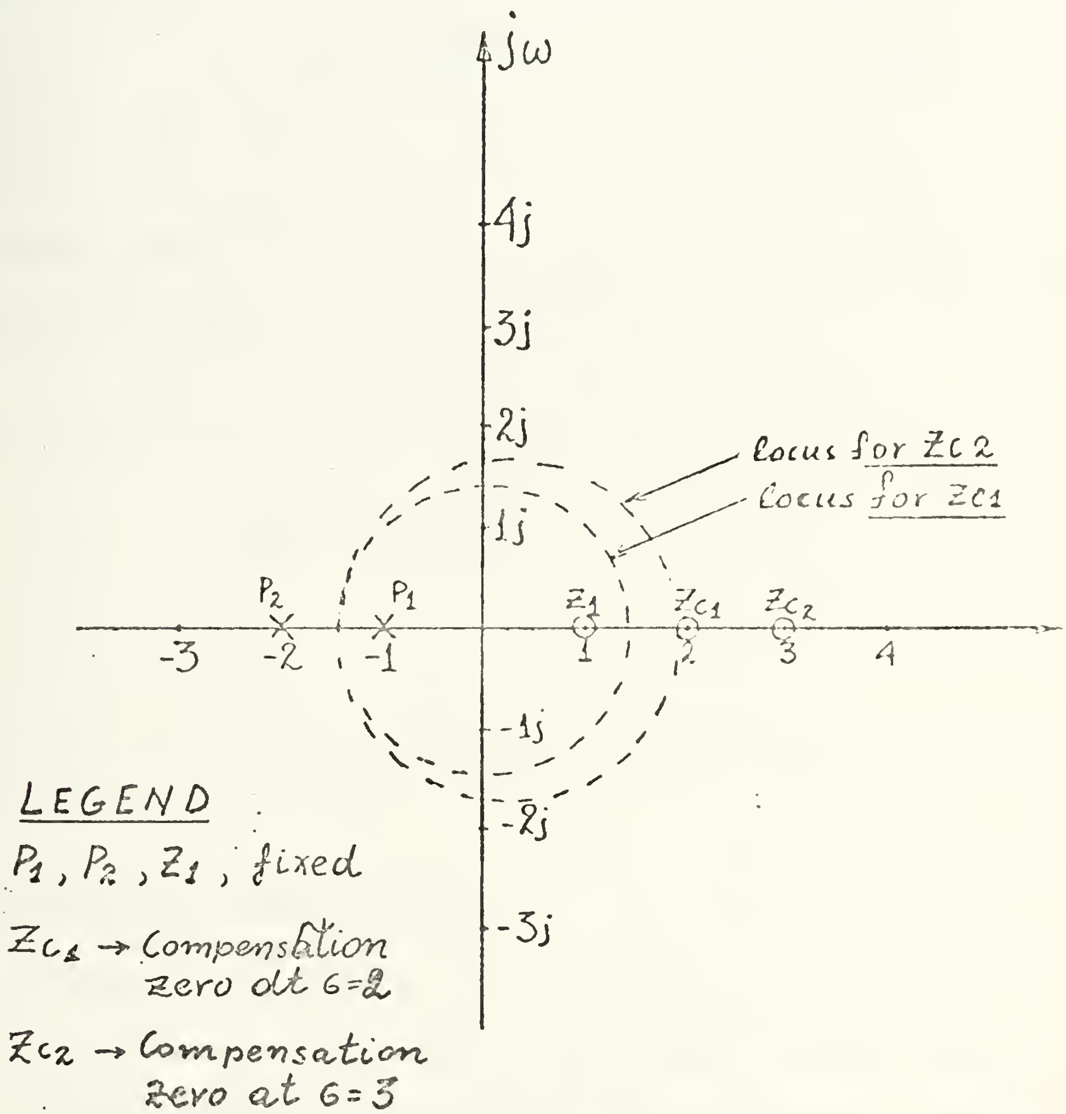


FIGURE 5-7

6. In figure 5-8, the results of the parameter space analysis are shown. In order to show that the results are valid, select the parameter space loci for ($\omega_n = 1.0$, $\zeta = 0.7$), represented by its two images (the dash-dotted lines). An arbitrary set of values for this loci is,

$$(A = 8.375,\ B = 2.32,\ \Gamma = 12.0),$$

which gives a second order polynomial, (equation V-4):

$$s^2 + \frac{29.625}{21.375}s + \frac{20.680}{21.375} = 0$$

with:

$$\omega_n = .985$$

$$\zeta = .706$$

For a second demonstration consider the loci for ($\omega_n = 1.6$, $\zeta = .6$). An arbitrary set of values for A, B, Γ on this loci is,

$$(A = -.718,\ B = -3.5,\ \Gamma = 3.0),$$

giving a second order polynomial,(equation V-4):

$$s^2 + \frac{6.218}{3.282}s + \frac{8.500}{3.282} = 0$$

with:

$$\omega_n = 1.62$$

$$\zeta = .603$$

These results turn out to be fairly accurate and the theory presented works in practice.

E. SOLUTION TO THE GENERAL PROBLEM

1. Consider again equation (V-2). The discussion in section (V-B) indicated the necessity for an infinite number of cuts to be done for the complete parameter plane solution in three dimensions. However, depending on the desired degree of accuracy a finite number of cuts will suffice for practical purposes.

2. The solution for a case with more than two cuts will be demonstrated in section 5.

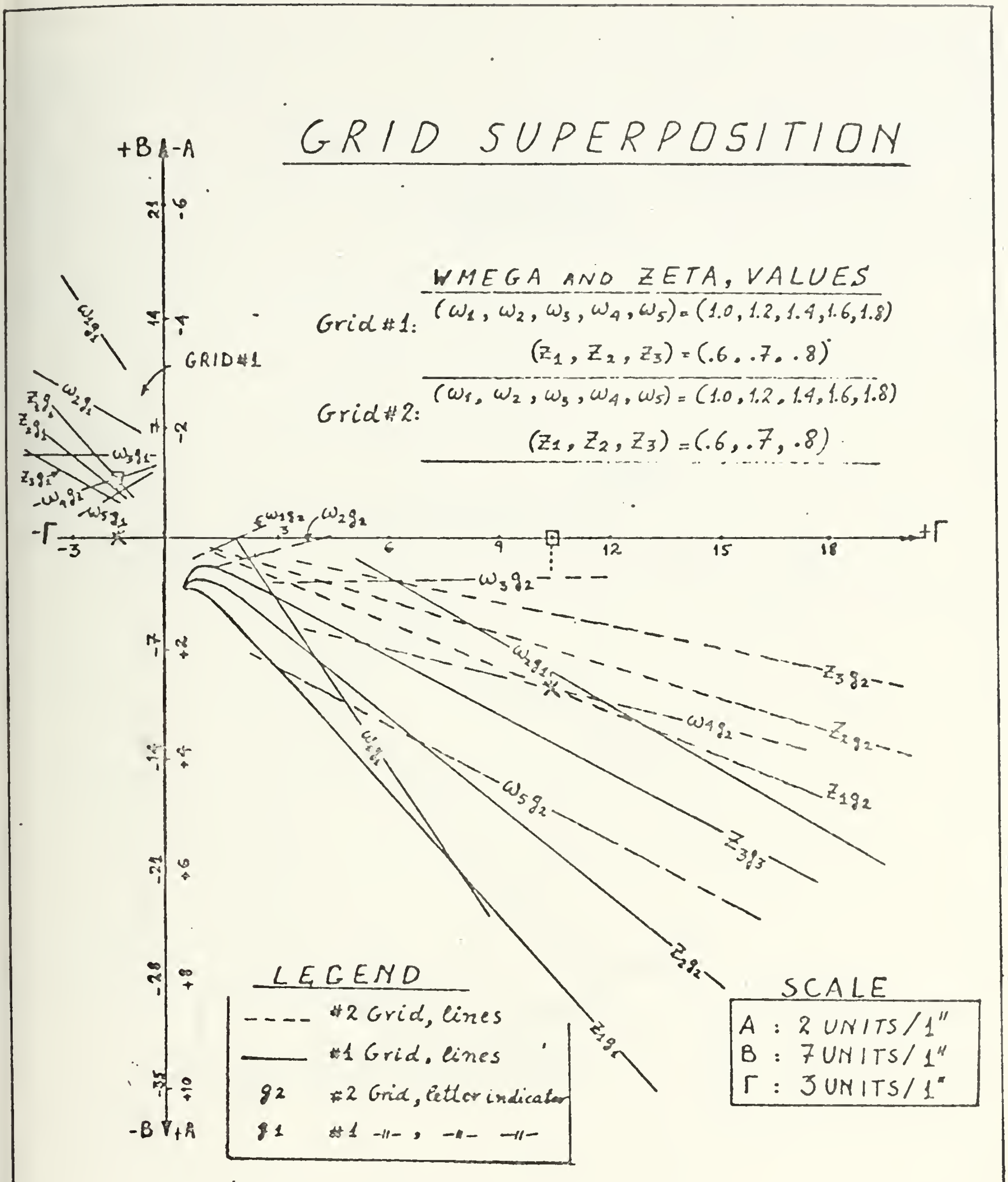

GRID SUPERPOSITION
WMEGA AND ZETA, VALUES
Grid #1: (ω1, ω2, ω3, ω4, ω5) = (1.0, 1.2, 1.4, 1.6, 1.8)
(Z1, Z2, Z3) = (.6, .7, .8)
Grid #2: (ω1, ω2, ω3, ω4, ω5) = (1.0, 1.2, 1.4, 1.6, 1.8)
(Z1, Z2, Z3) = (.6, .7, .8)
GRID #1
+B -A
-B +A
-Γ +Γ
LEGEND
---- #2 Grid, lines
—— #1 Grid, lines
g2 #2 Grid, letter indicator
g1 #1 -"- , -"- -"-
SCALE
A : 2 UNITS/1"
B : 7 UNITS/1"
Γ : 3 UNITS/1"

FIGURE 5-8a

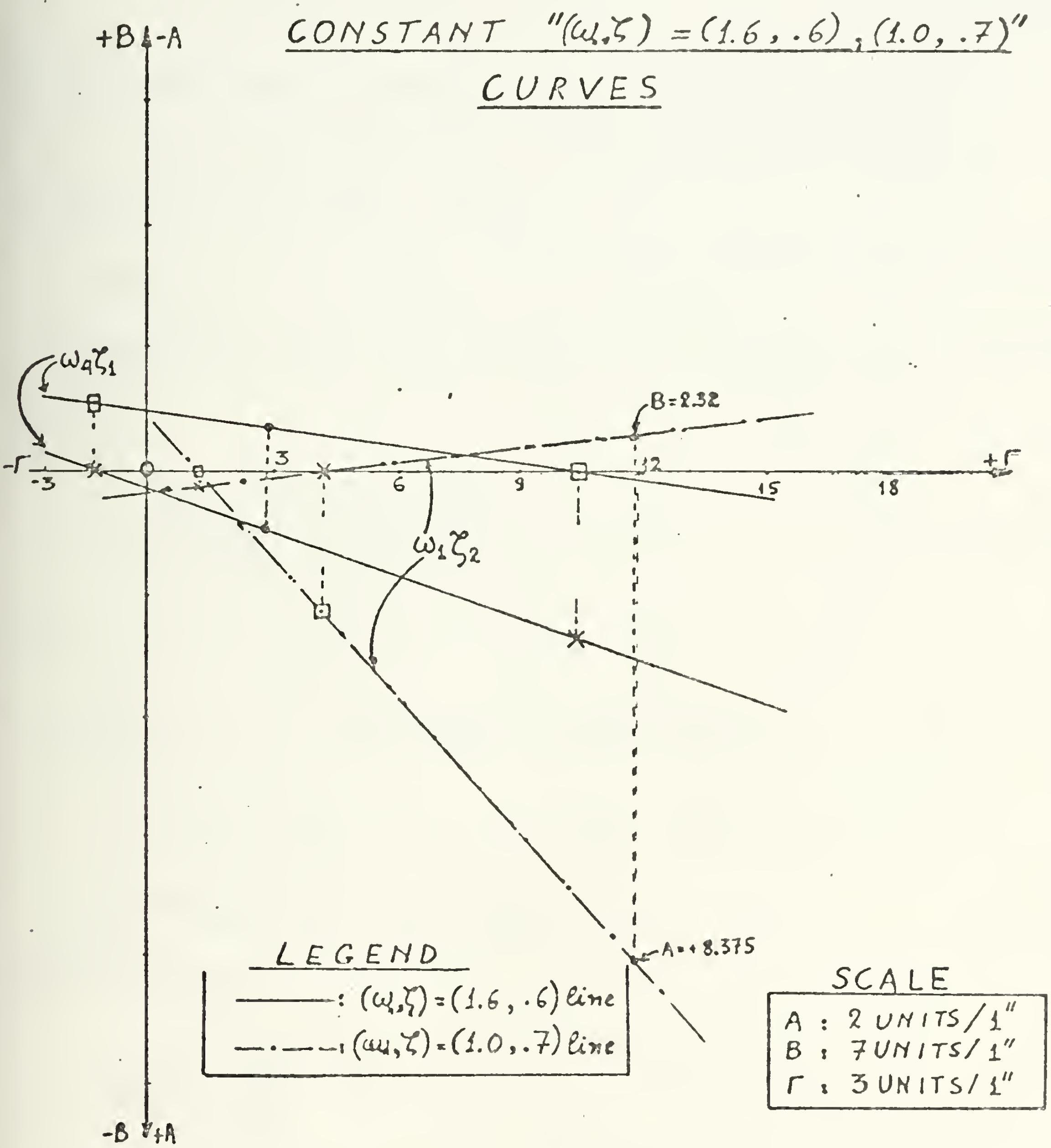

CONSTANT "(ω,ζ) = (1.6, .6), (1.0, .7)"
CURVES
+B
-A
-B
+A
-Γ
+Γ
-3
3
6
9
12
15
18
$\omega_a\zeta_1$
$\omega_1\zeta_2$
B=2.32
A=+8.375
LEGEND
: (ω,ζ) = (1.6, .6) line
: (ω,ζ) = (1.0, .7) line
SCALE
A : 2 UNITS/1"
B : 7 UNITS/1"
Γ : 3 UNITS/1"

FIGURE 5-8b

3. The equations for the general cut through the Γ-axis will be derived by letting:

$$A = \lambda B \tag{V-5}$$

as the plane of the cut implies.

4. By substituting equation (V-5) into equation (V-1), the following set of two equations results:

$$B(\lambda\alpha_1 + \delta_1) + \Gamma\epsilon_1 + B^2\lambda\phi_1 + B\Gamma(\lambda\eta_1 + \varphi_1) + B^2\Gamma\lambda\mu_1 = -\xi_1 \tag{V-5a}$$

$$B(\lambda\alpha_2 + \delta_2) + \Gamma\epsilon_2 + B^2\lambda\phi_2 + B\Gamma(\lambda\eta_2 + \varphi_2) + B^2\Gamma\lambda\mu_2 = -\xi_2 \tag{V-5b}$$

Therefore:

$$\Gamma = \frac{-\xi_1 - (\lambda\alpha_1 + \delta_1)B - \lambda\phi_1 B^2}{\epsilon_1 + B(\lambda\eta_1 + \varphi_1) + B^2\lambda\mu_1} \tag{V-6a}$$

and:

$$\begin{aligned} &B^4\lambda^2\Delta_{\mu\phi} + B^3\left[\lambda^2(\Delta_{\mu\phi} + \Delta_{\eta\phi}) + \lambda(\Delta_{\mu\delta} + \Delta_{\varphi\phi})\right] + \\ &B^2\left[\lambda^2\Delta_{\eta\alpha} + \lambda(\Delta_{\varphi\alpha} + \Delta_{\eta\delta} + \Delta_{\epsilon\phi} + \Delta_{\mu\xi}) + \Delta_{\varphi\delta}\right] + \\ &B\left[\lambda(\Delta_{\epsilon\alpha} + \Delta_{\eta\xi}) + (\Delta_{\varphi\xi} + \Delta_{\epsilon\delta})\right] + \Delta_{\epsilon\xi} = 0 \end{aligned} \tag{V-6b}$$

where in general:

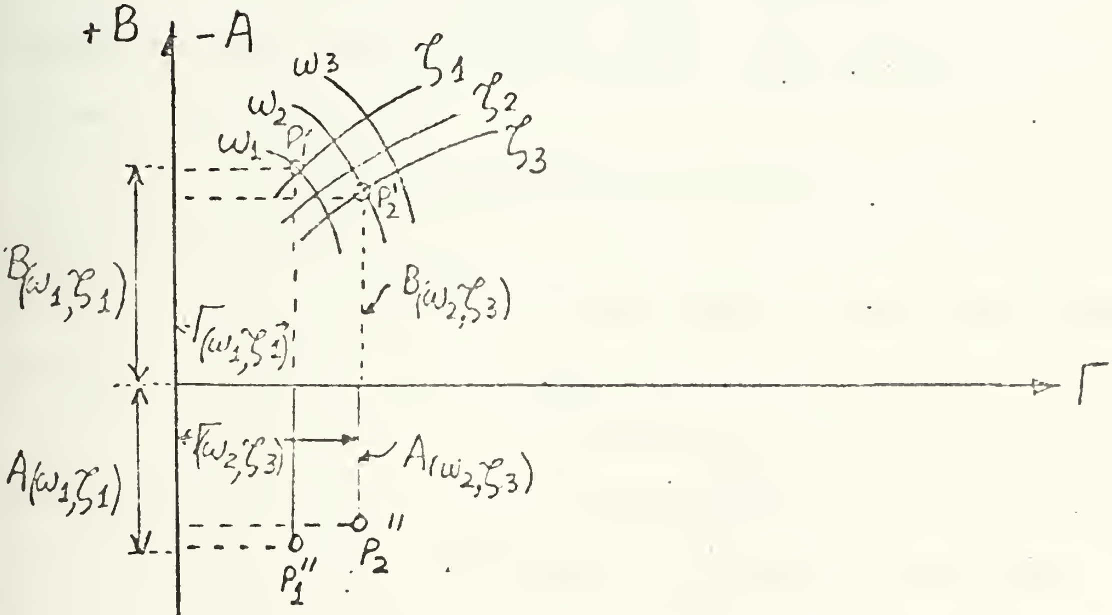
$$\Delta_{xy} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1y_2 - y_1x_2$$

5. So in order to define a certain desired "ω_t, ζ_t" point in the A,B,Γ space, one should follow steps as follows:

a. Define the coefficients $\alpha_1, \beta_1, \ldots, \xi_1$ and $\alpha_2, \beta_2, \ldots \xi_2$, which become constants for $\omega = \omega_t$, $\zeta = \zeta_t$. It is assumed of course that the time delay in the system is fixed.

b. Solve for B from the equation (V-6b). A digital computer

CONSTRUCTION OF THE "SECOND PROJECTION" FOR A CERTAIN GRID, BY USE OF THE CUT-INDEX "λ".



NOTE

$$A(\omega_1,\zeta_1) = \lambda \cdot B(\omega_1,\zeta_1)$$

$$A(\omega_2,\zeta_3) = \lambda \cdot B(\omega_2,\zeta_3)$$

FIGURE 5-9

routine will be necessary for this purpose. However, the conventional solution of a quartic equation is listed as Appendix E, for easier reference. Care should be taken to exclude any complex roots.

c. Then Γ will be given by using (V-6a), for each admissible B-root.

d. If track of the cut-plane is kept the other projection is not necessary, since $A = \lambda B$ (V-5), and therefore for each point in the $B\Gamma$ grid that will be generated by the procedure of subparagraphs 5a 5c, there will correspond another one as shown in figure 5-9.

In figure 5-10a the grids of constant ζ and ω_n curves obtained for three different cuts are shown. Notice that one projection for each cut is sufficient.

To construct an approximate constant ω_n, ζ_n curve one must select the points P_1', P_2', P_3' on the corresponding grids (one from each available grid), and by joining them by straight line segments obtains one of the two required approximate projections.

In figure 10-b this projection is shown, with the construction grid eliminated. The second projection of the constant ω_n, ζ_n curve can be approximately defined, by locating the projections P_1'', P_2'', P_3'' from equation (V-5), with $\lambda = \lambda_1, \lambda_2, \lambda_3$, etc.

F. EXAMPLE ON A SIMPLE DESIGN PROBLEM

Consider the hypothetical system of figure 5-11, and suppose that the operational requirements of the system are met if the operating point is $\omega_\alpha = .4$ rad/sec and $\zeta_\alpha = .5$.

Consider also that the cost for building this system was found to be $C = M \cdot \Gamma$ where M is a big number and T is so small that $e^{-Ts} = 1$ (in order to simplify the algebra). There exists no information about

CURVED "$(\omega_h, \zeta_h) = $ CONST."-LINE, FROM GRID SUPERPOSITION

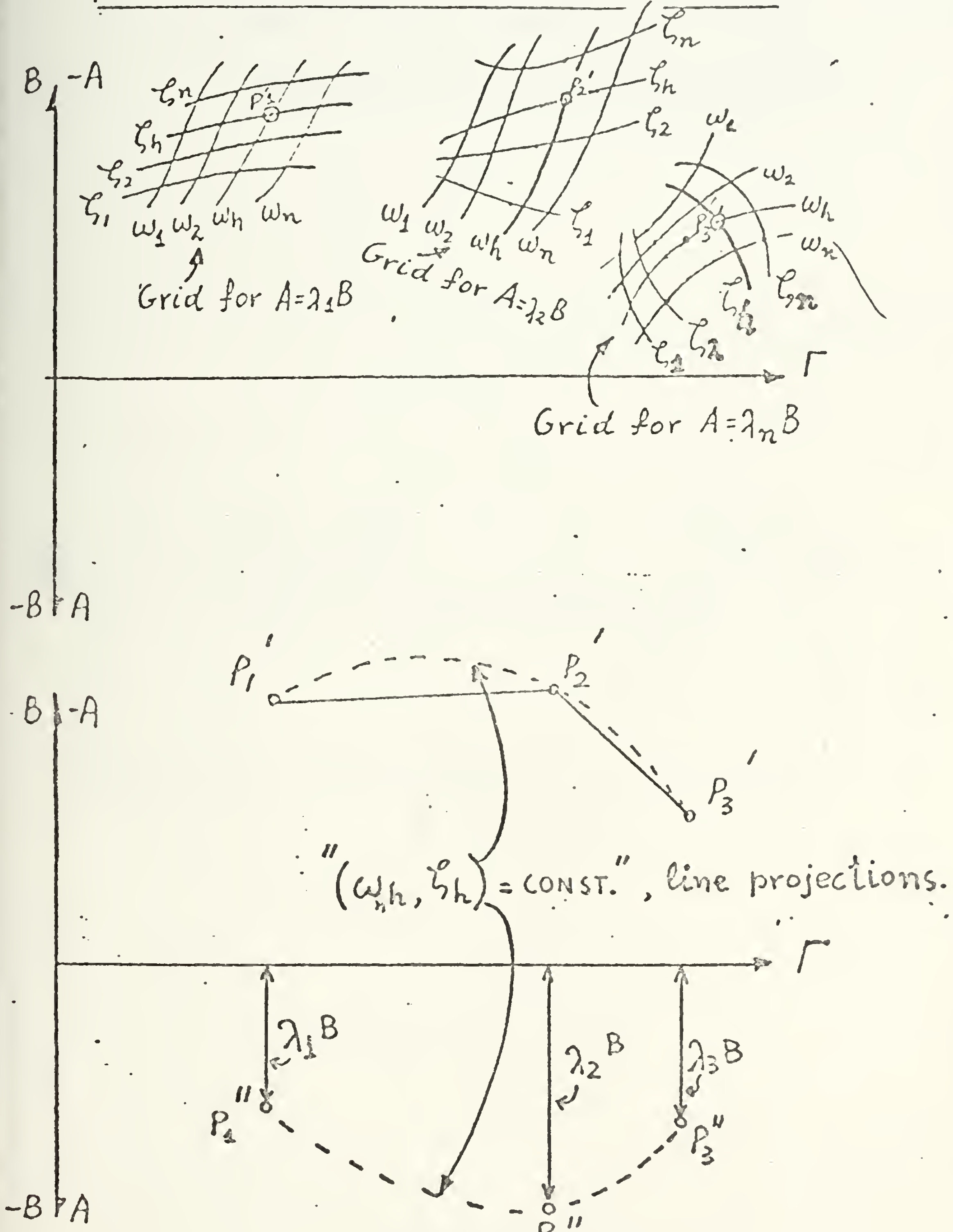


FIGURE 5-10

BLOCK DIAGRAM FOR A SYSTEM
WITH TIME-DELAY
(PRODUCTS INCLUDED)

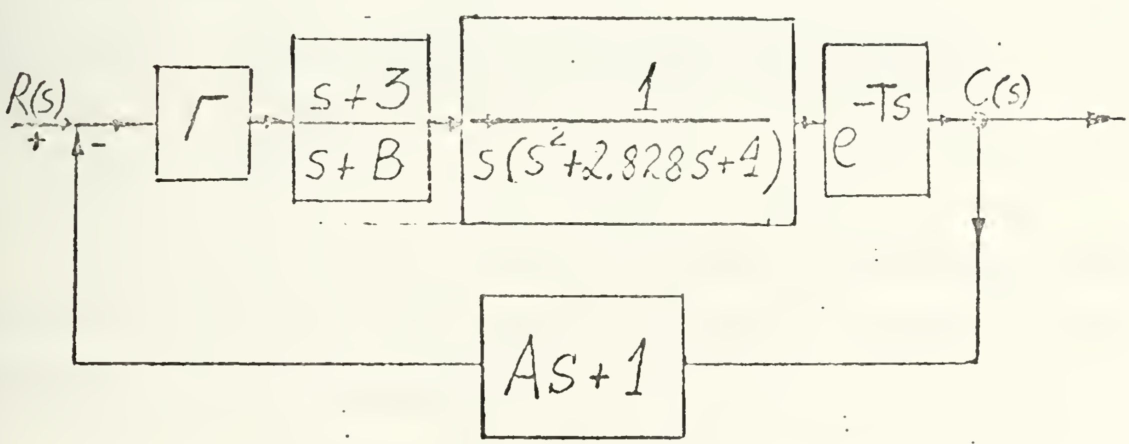


FIGURE 5-11

the operating cost of the system as a function of A, B, Γ, so this aspect will not be considered, but of course the stability is presupposed.

It is desired to select A, B, Γ that will simultaneously give ω_α, ζ_α as specified and minimum possible cost.

It is noted that A physically represents the positive gain of an amplifier, therefore there exists the restriction $A > 0$.

SOLUTION

1. The closed loop gain of the system turns out to be:

$$\frac{C}{R} = \frac{\Gamma(s+3)(As+1)e^{-Ts}}{\left[s^4 + (2.828+B)s^3 + (4+2.828B)s^2 + 4Bs\right] + e^{-Ts}\left[A\Gamma(s^2+3s) + \Gamma(s+3)\right]}$$

So the characteristic equation for the general cut-plane $A = \lambda B$, and $T \to 0$ will become:

$$CE(s) = \left[s^4 + 2.828s^3 + 4s^2\right] + B\left[s^3 + 2.828s^2 + 4s\right] + \Gamma\left[s+3\right] + B\Gamma\left[\lambda s^2 + 3\lambda s\right] = 0.$$

2. At this point one must make a decision on selecting the number of cuts that will suffice the specific application. For this case consider eight cuts, corresponding to:

$$\lambda = 0,\ \lambda = \pm\tfrac{1}{2},\ \lambda = \pm 1,\ \lambda = \pm 2,\ \lambda = \infty$$

In table 5-1 the resulting CE(s) are tabulated.

3. By applying the method of section VE, the curve for constant ($\omega_n = .4$ and $\zeta = .5$), was obtained and displayed in figures 5-12a, 5-12b. These figures have been plotted according to the data of table 5-2.

In figure 5-12a the interesting situation for the stable compensation of the system under design is displayed.

4. According to the available information:

a. Gain Upper Limit: $\Gamma=0.217$ units (stable operation on specified ω_n, ζ).

TABLE 5-1

λ	TYPE OF COEFFICIENT	DECREASING-POWER, TERMS → s^4	s^3	s^2	s^1	s^0
0	CONSTANT	1.0	2.828	4.0	0.0	0.0
	B	0.0	1.0	2.828	4.0	0.0
	Γ	0.0	0.0	0.0	1.0	3.0
	B Γ	0.0	0.0	0.0	0.0	0.0
1/2	CONSTANT, B, Γ	SAME AS FOR $\lambda = 0$				
	B Γ	0.0	0.0	0.5	1.5	0.0
-1/2	CONSTANT, B, Γ	SAME AS FOR $\lambda = 0$				
	B Γ	0.0	0.0	-0.5	-1.5	0.0
1	CONSTANT, B, Γ	SAME AS FOR $\lambda = 0$				
	B Γ	0.0	0.0	1.0	3.0	0.0
-1	CONSTANT, B, Γ	SAME AS FOR $\lambda = 0$				
	B Γ	0.0	0.0	-1.0	-3.0	0.0
2	CONSTANT, B, Γ	SAME AS FOR $\lambda = 0$				
	B Γ	0.0	0.0	2.0	6.0	0.0
-2	CONSTANT, B, Γ	SAME AS FOR $\lambda = 0$				
	B Γ	0.0	0.0	-2.0	-6.0	0.0
∞	CONSTANT	1.0	2.828	4.0	0.0	0.0
	A	0.0	0.0	0.0	0.0	0.0
	Γ	0.0	0.0	0.0	1.0	3.0
	A Γ	0.0	0.0	1.0	3.0	0.0

CONTINUATION TO TABLE 5-1

	TYPE OF COEFFICIENT	DECREASING-POWER, TERMS →				
		s^4	s^3	s^2	s^1	s^0
	CONSTANT	1.0	2.828	4.0	0.0	0.0
1/100	B	0.0	1.0	2.828	4.0	0.0
	Γ	0.0	0.0	0.0	1.0	3.0
	B Γ	0.0	0.0	1/1000	3/1000	0.0
	CONSTANT	SAME AS FOR $\lambda = .001$, (AND $\lambda = 0$)				
-1/100	B					
	Γ					
	B Γ	0.0	0.0	-1/1000	-3/1000	0.0
	CONSTANT	SAME AS FOR $\lambda = .001$, (AND $\lambda = 0$)				
/100	B					
	Γ					
	B Γ	0.0	0.0	1000	3000	0.0
	CONSTANT	SAME AS FOR $\lambda = .001$, (AND $\lambda = 0$)				
-000	B					
	Γ					
	B Γ	0.0	0.0	-1000	-3000	0.0

TABLE 5-2

λ	ω_n	ℓ	$\Gamma_{(+)}$	$A_{(+)}$	$B_{(+)}$	$\Gamma_{(-)}$	$A_{(-)}$	$B_{(-)}$	REMARKS ON $A, B, \Gamma(+)$	ON $A, B, \Gamma(-)$
$\frac{1}{2}$	.402	.5	.212	.212	.424	-2.287	-17.667	-35.333	STABLE	UNSTABLE SET
$-\frac{1}{2}$	.402	.5	.218	-.257	.514	2.222	-14.587	29.173	UNSTABLE	— " —
1	.402	.5	.210	.391	.391	-1.156	-19.151	-19.151	STABLE	— " —
-1	.402	.5	.223	-.578	.578	1.089	-12.959	12.959	UNSTABLE	— " —
2	.402	.5	.206	.680	.340	-.588	-.220	-.110	STABLE	— " —
-2	.404	.5	.242	-1.638	.819	.506	-9.120	4.560	UNSTABLE	— " —
$\frac{1}{1000}$	.404	.5	.217	.000467	.467	-1127.8	-159.79	-15979	STABLE	— " —
$-\frac{1}{1000}$	.404	.5	.217	-.000476	.476	1127.7	-159.73	15973	UNSTABLE	— " —
1000	.404	.5	.184	2.84	.00284	-.00133	-2629.	-2.629	STABLE	— " —
-1000	.404	.5	.00133	2592.	-2.592	.184	2.88	-.00288	UNSTABLE	— " —

b. Gain Lower Limit: $\Gamma > 0.0$. Nevertheless it was possible to get a minimum gain of .184 units for the lowest B parameter employed.

c. Compensation Pole:

$0.00284 < B < 0.47$ in the range $\Gamma = (0.184 \div 0.217)$

d. Tachometer Feed-Back:

$0.000467 < A < 2.84$ in the same Γ range. So it follows that the system is more sensitive in "B" than in "A" variations (for this specific Γ range).

5. So in order to satisfy the requirement of as small gain as possible, imposed by the specifications and having the information of figure 5-12a, one should choose of course the set:

$$(A, B, \Gamma)_1 = (2.84, .00284, .184).$$

But if it was possible to pay 9 percent more*4 the solution

$$(A, B, \Gamma)_2 = (.115, .25, .20)$$

is much more advantageous and takes care of possible component drift.

6. By substituting the set $(A, B, \Gamma) = (.115, .25, \Gamma)$ in CE(s), (for T=0), one gets:

$$CE(s) = \left[s^4 + 3.078s^3 + 4.707\ s^2 + 1.00s\right] + \Gamma\left[1.15s^2 + 4.445s + 3\right] = 0.$$

the root-locus plot of which gives $(\omega_n, \zeta)_{dominant} = (.3986, .477)$ for $\Gamma = .2$, which is very close to the desired specifications (see figure 5-12c).

7. It is of interest to note what the constant $\sigma = 0$ and $\zeta = 0$ surfaces look like.

a. From $CE(\sigma)$ for $s = (\sigma \equiv 0)$ the $\sigma = 0$ surface is the $\Gamma = 0$ plane.

---

*4 Since the cost function is $C = M\Gamma$ it follows that $\frac{\Delta c}{c_1} = \frac{\Gamma_2 - \Gamma_1}{\Gamma_1} = .087$

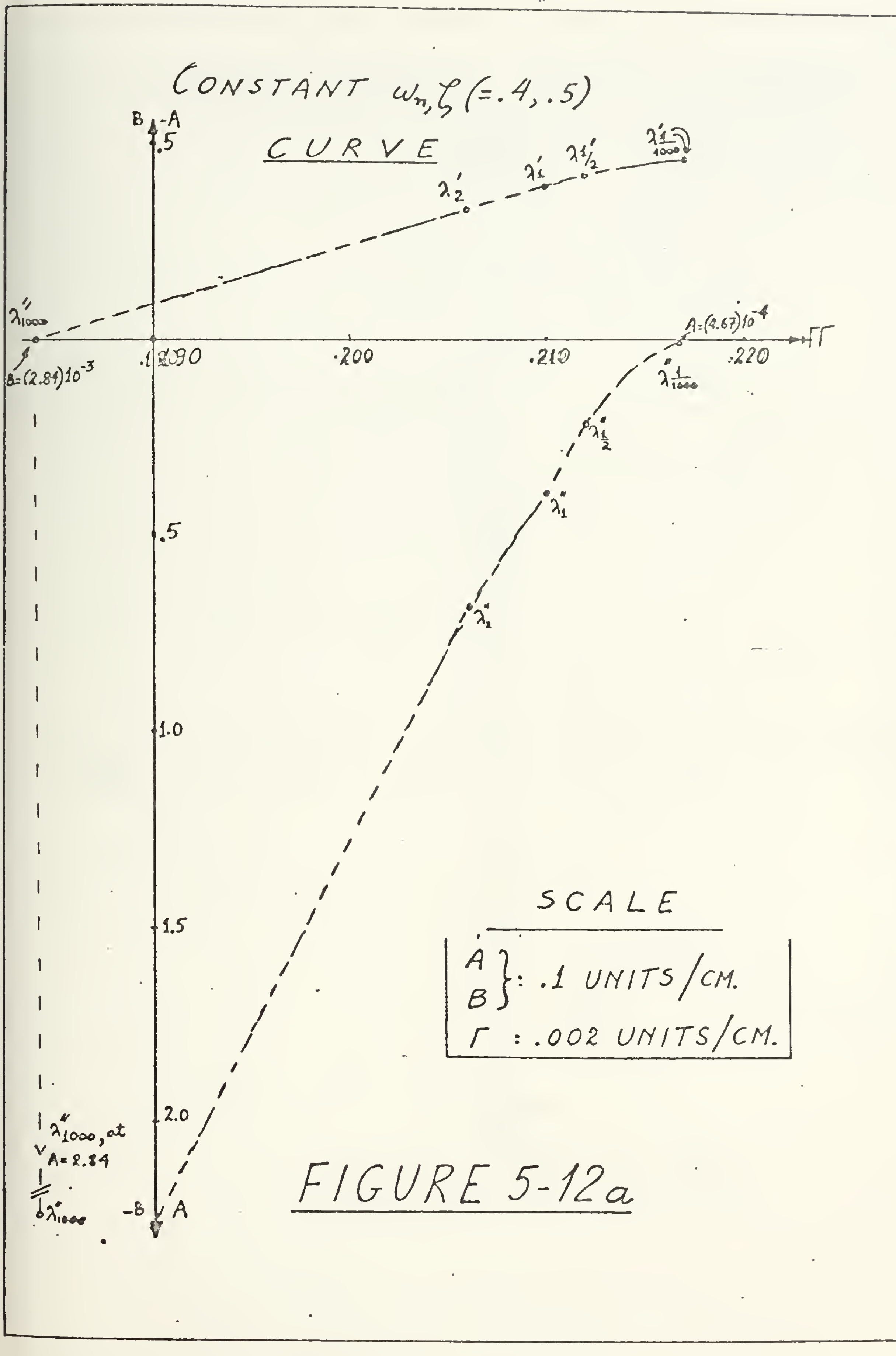

CONSTANT ω_n, ζ (=.4, .5)
CURVE
B -A
.5
λ'_2
λ'_1
$\lambda'_{1/2}$
$\lambda'_{1/1000}$
λ''_{1000}
B=(2.84)10^{-3}
.190
.200
.210
.220
A=(4.67)10^{-4}
Γ
$\lambda''_{1/1000}$
$\lambda''_{1/2}$
λ''_1
λ''_2
.5
1.0
1.5
2.0
λ''_{1000}, at A=2.84
λ''_{1000}
-B
A
SCALE
A, B : .1 UNITS/CM.
Γ : .002 UNITS/CM.

FIGURE 5-12a

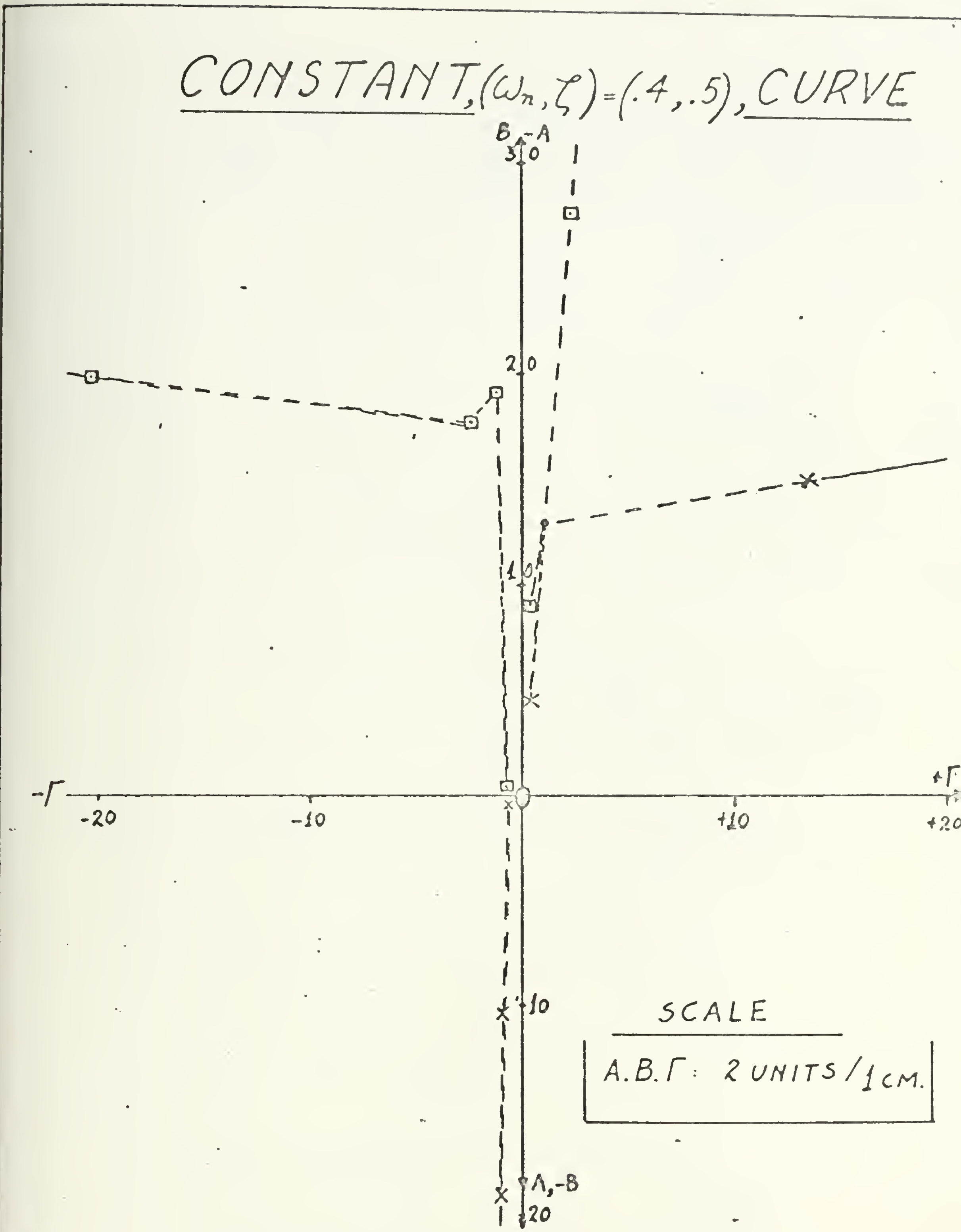

CONSTANT, $(\omega_n, \zeta) = (.4, .5)$, CURVE
B, -A
30
20
10
10
A, -B
20
$-\Gamma$
$+\Gamma$
-20
-10
+10
+20
SCALE
A.B.Γ: 2 UNITS/1 CM.

FIGURE 5-12b

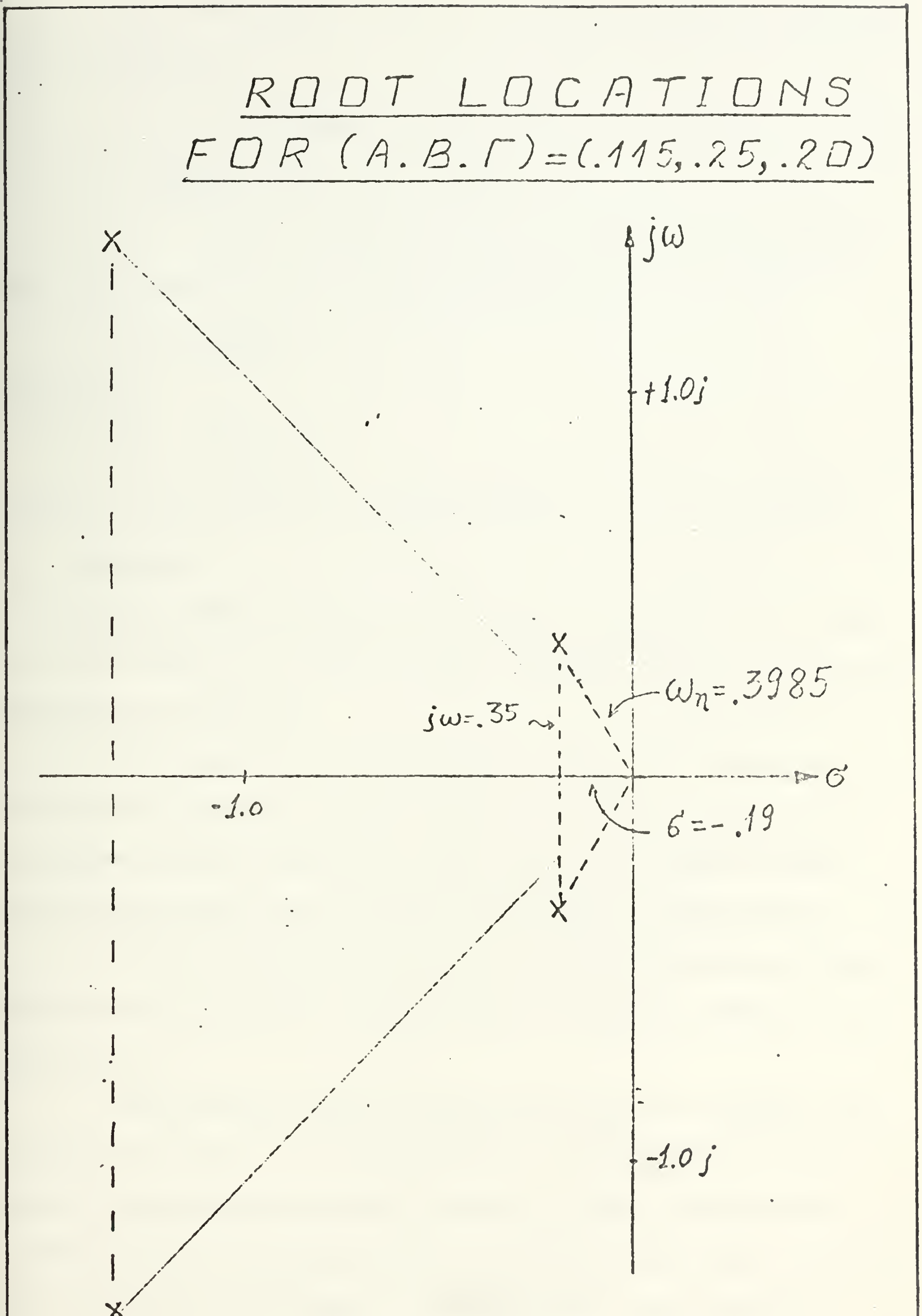

ROOT LOCATIONS
FOR (A.B.Γ)=(.115,.25,.20)
jω
+1.0j
ωn=.3985
jω=.35
-1.0
σ
σ=-.19
-1.0j

FIGURE 5-12c

b. From $CE(j\omega)$ for the $s = j\omega$ surface one gets the following set of equations.

$$\omega^4 - (4 + 2.828B)\omega^2 - A\Gamma\omega^2 + 3\Gamma = 0$$

$$-(2.828 + B)\omega^3 + 4B\omega + 3A\Gamma\omega = 0$$

c. By simultaneously solving the $\sigma = 0$ and $\zeta = 0$ equations the equation for the cut of those two surfaces is:

$$B^2 + 2.828B + 4 = 0$$

which gives:

$$B = -\sqrt{2} \pm j\sqrt{2}$$

Therefore there exists no intersection of the $\sigma = 0$, $\zeta = 0$ surfaces.

G. CONCLUSIONS

From the theory presented in this section theoretically and shown to actually work for the simple system of figure 5-11, the following conclusions can be obtained:

1. It is possible to construct the constant "ω-ζ" curves for a three variable parameter system. Care must be taken in order not to exceed the physical limitations of the system, e.g. taking cuts that would perhaps destroy the feed-back or forward gain of the system.

2. In the case where the variable parameters show in a nonlinear fashion in the system, the constant "ω-ζ" curves have no straight line projections.

3. Care should be taken to differentiate between different segments of the solution. It is of importance to define which of them provide stable and which unstable operation. Until a well grounded stability theorem is developed in the three dimensional space, this task which was easy for the system of section (V-F) might prove impossible

for a complicated situation. In this case the method of this section will be handicapped.

4. Another thing to notice, is that for the nonlinear case, (A,B,Γ products), it is possible to come up with a curve which is more or less normal to the Γ-axis. If this happens it is not easy to read the appropriate A,B,Γ sets in one hand, but sensitivity on the Γ parameter is proved. So if one wants to operate under the conditions imposed by the curve, one must set $\Gamma = \Gamma_o$ and then possibly one can reduce the problem to a two dimensional one.

5. Sensitivity analysis of a system is possible for a multivariable control--system, by applying methods described in this section. This is in contrast to the approach of using a family of root loci where the method of sensitivity analysis is not so direct.

6. Another great advantage of the parameter space approach over the root loci method is that system optimization through adjustment of parameters is more direct, and illustrative.

7. Of course in order to arrive at the desired result a vast amount of analysis was necessary, and it is a disadvantage which of course is removed when a computer program similar to the one in Appendix D is available. So indeed, the method is "digital-computer" oriented.

VI. AN ACTUAL ENGINEERING APPLICATION

A. GENERAL

1. In this section the roll angle control system for a vertical take-off aircraft is studied.

2. In a post-design analysis of this system, (which was initially designed by the root-locus methods and thereafter operated successfully), attempts to locate the actual operating point inside the two variable parameter stability region were originally unsuccessful. This implied that the stability criterion of chapter IV was invalid, and since the derivations of the stability criterion was rigorous, further study of the particular case is needed.

3. In the subsections to follow, the system is first considered from an elementary aspect and since it needs compensation, its possible modes are defined.

Then a parameter plane design follows and having in mind the results of the stability analysis, an appropriate operating point is selected. Time domain simulations follow at last to insure the correctness of the parameter plane results.

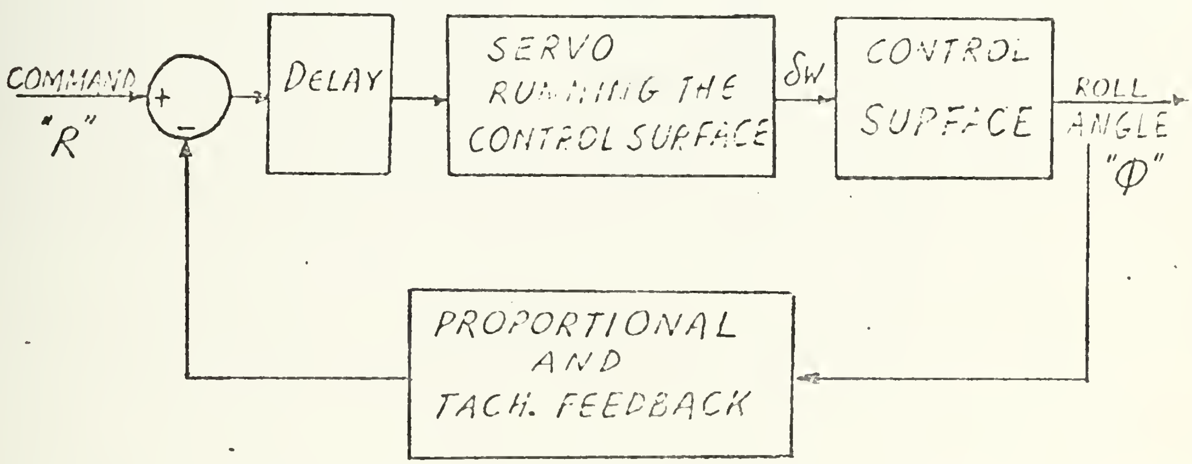
B. THE SYSTEM

1. Brief Physical Description

a. The system is presented in block diagram form in figure 6-1.

In the component analysis of this figure one can observe seven variable parameters. But since the servo constants have been preset to their values of table 6-1, the only variable parameters remaining are K_p, K_ϕ, and T (which is bounded within limits).

BLOCK DIAGRAM FOR THE ROLL-ANGLE CONTROL SYSTEM OF A VERTICAL TAKE OFF A/C..



COMPONENT ANALYSIS

DELAY $= e^{-Ts}$

SERVO $= K/Ds^2 + Es + F$

C. SURFACE TRANSFER FUNCTION:

$$\frac{\Phi}{\delta W} = \frac{(.305)\left[S^2+2(.318)(.698)S+(.698)^2\right]}{(S-.0434)(S+1.464)\left[S^2+2(.243)(.864)S+(.864)^2\right]}$$

FEED-BACK $= (K_p \cdot S + K_\phi)$

FIGURE 6-1

TABLE 6-1

VALUES OF CONSTANTS USED IN THE SYSTEM OF FIGURE 6-1

CONSTANT	VALUE
T	.15 ÷ .30
K	1/8.1 = .1235
D	$.488 \times 10^{-3}$
E	$.4719 \times 10^{-1}$
F	2.329

b. Since the permissible range of variation on T is small, it is reasonable to start with the minimum expected value of T, (0.15), since the results of section IV show that this should give the maximum stability region in the A,B plane.

2. Objectives

a. To actually define appropriate K_p, K_ϕ values for stable operation. In doing so one should have in mind that zetas around 0.7 and frequencies from 1 to 3 rad/sec are desirable from a "pilot-performance" point of view.*5

b. To check and find if the operating values of K_p, K_ϕ lie in the parameter plane stability region in order to check on the discrepancy reported (section VI A). These values are:

$$(K_p, K_\phi) = (95, 190)$$

3. Systems Transfer Characteristic

$$\frac{\phi}{R}(s) = \frac{(.305)\frac{K}{D}e^{-Ts}\left[s^2 + .444s + .488\right]}{F(s) + e^{-Ts}\left[A\cdot P_A(s) + B\cdot P_B(s)\right]} \qquad \text{(VI-1)}$$

where:

$$A\cdot P_A(s) = A\left[s^3 + .444s^2 + .488s + 0.0\right]$$

$$B\cdot P_B(s) = B\left[0\cdot s^3 + s^2 + .444s + .488\right]$$

and:

$$A = \frac{(.305)K}{D}K_p = 77.2\ K_p \quad \Rightarrow \quad \text{Tachometer}$$

$$B = \frac{(.305)K}{D}K_\phi = 77.2\ K_\phi \quad \Rightarrow \quad \text{Proportional Control}$$

$$F(s) = s^6 + 98.4696s^5 + 4941.1680s^4 + 8978.082s^3 + 5086.9961s^2 + 3162.6011s - 147.59$$

---

*5 These were established by the original system designer.

4. <u>The Systems Characteristic Equation is:</u>

$$CE(s) = \left[s^6 + 98.4696s^5 + 4941.1680s^4 + 8978.082s^3 + 5086.9961s^2 + 3162.6011s - 147.59\right] + e^{-Ts} A\left[s^3 + .444s^2 + .488s + 0.0\right] + e^{-Ts} B\left[0 \cdot s^3 + s^2 + .444s + .488\right] = 0 \qquad \text{(VI-3)}$$

C. POSSIBLE COMPENSATION MODES

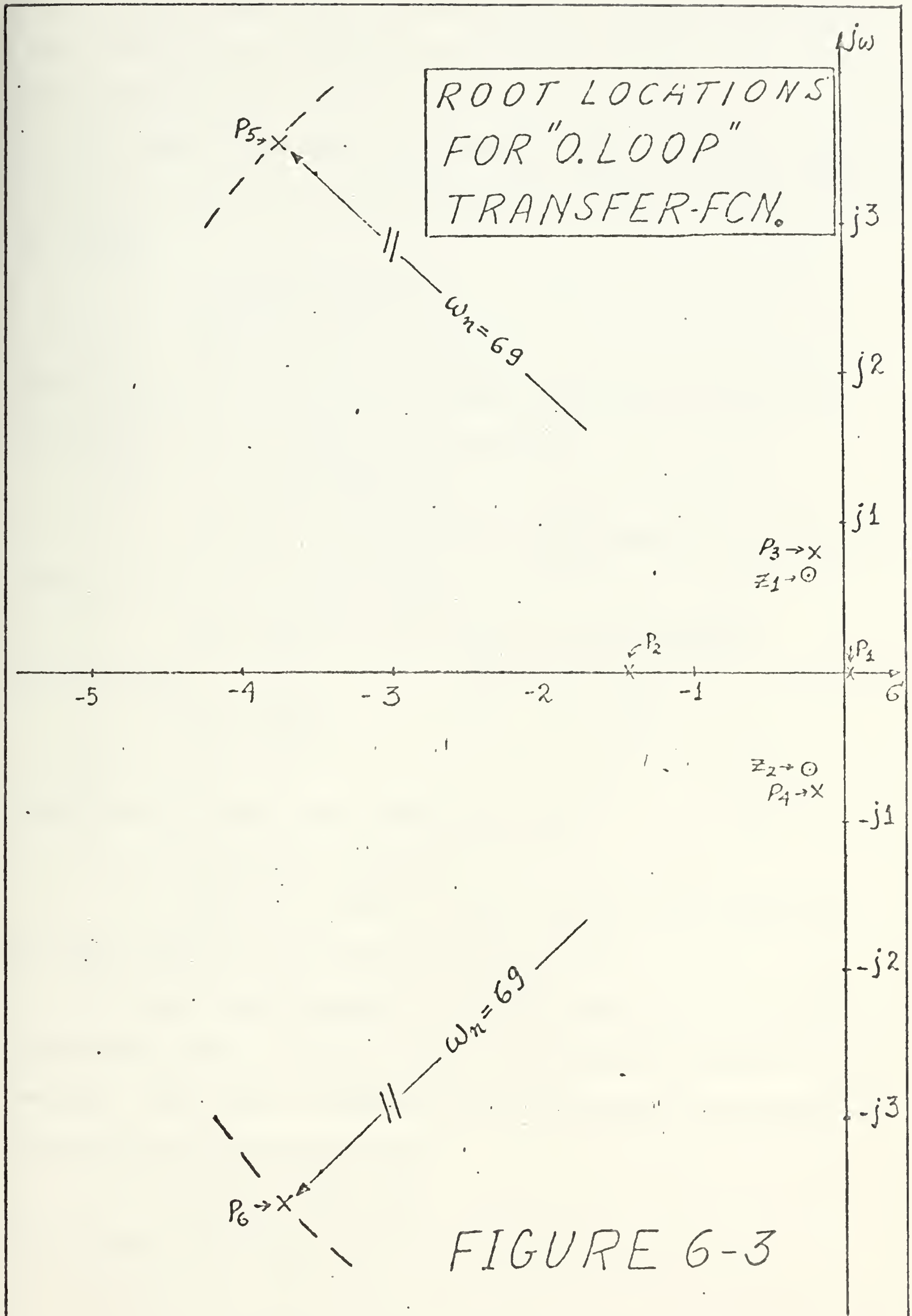
1. By looking closer to the feed-back component of the figure 6-1 system, we can see in general that its effect on the open loop transfer function is the insertion of a compensation zero "z_c", and by controlling the values of the variable parameters K_p and K_ϕ one can cause various compensation modes.

2. Noting also the results of section II, where it was observed that though the shape of a root-locus was symmetrically deformed towards the $j\omega$-axis, its basic form remained unchanged with varying delay, <u>one can search for the modes mentioned above</u> using the delay-free version of figure 6-1 without loss of generality.

3. In figure 6-3 the open-loop root locations of the system are shown, with compensation z_c not included.

So with no compensation the pole P_1 will produce instability in the desired operating range.

Note also: first, that poles P_5 and P_6 located at a natural frequency contour of 69 rads/sec with a $\zeta = .7$, are too far away to be considered at the desired frequency range and second, that zeroes z_1, z_2 restrict root-travel branches leaving poles P_3, P_4, to remain in a very limited range in the s-plane, so that they force the continuous presence of a complex root pair with a set of $(\zeta, \omega_n) = (.43, .7)$ on the average. Theoretical cancellation would of course happen for infinite



ROOT LOCATIONS FOR "O.LOOP" TRANSFER-FCN.
jω
j3
j2
j1
-j1
-j2
-j3
P5
P6
P3
P4
P2
P1
Z1
Z2
ωn=69
ωn=69
-5
-4
-3
-2
-1
σ
FIGURE 6-3

gain, but since these four roots are very close together in pole-zero pairs, the influence of (P_3, P_4) in the systems transient characteristic will be much reduced.

4. Now the real zero "z_c" introduced by the feed-back, can generally take three possible relative locations with respect to the P_1, P_2 pair, as follows:

a. z_c to the Left of P_2.

This will cause a closed form of the root locus and for a proper gain will provide a pair of complex roots in the L.H.P. solely. In this case, both K_p and K_ϕ must be positive.

b. z_c to the Right of P_1.

In this case one of the two feed-back gains must be negative. So if in the feed-back expression

$K_p s + K_\phi$, K_p is factored out of the parentheses

one will have:

$$K_p \left(s + \frac{K_\phi}{K_p}\right)$$

which implies a negative gain locus, for $K_p < 0$.

c. z_c Between P_2 and P_1.

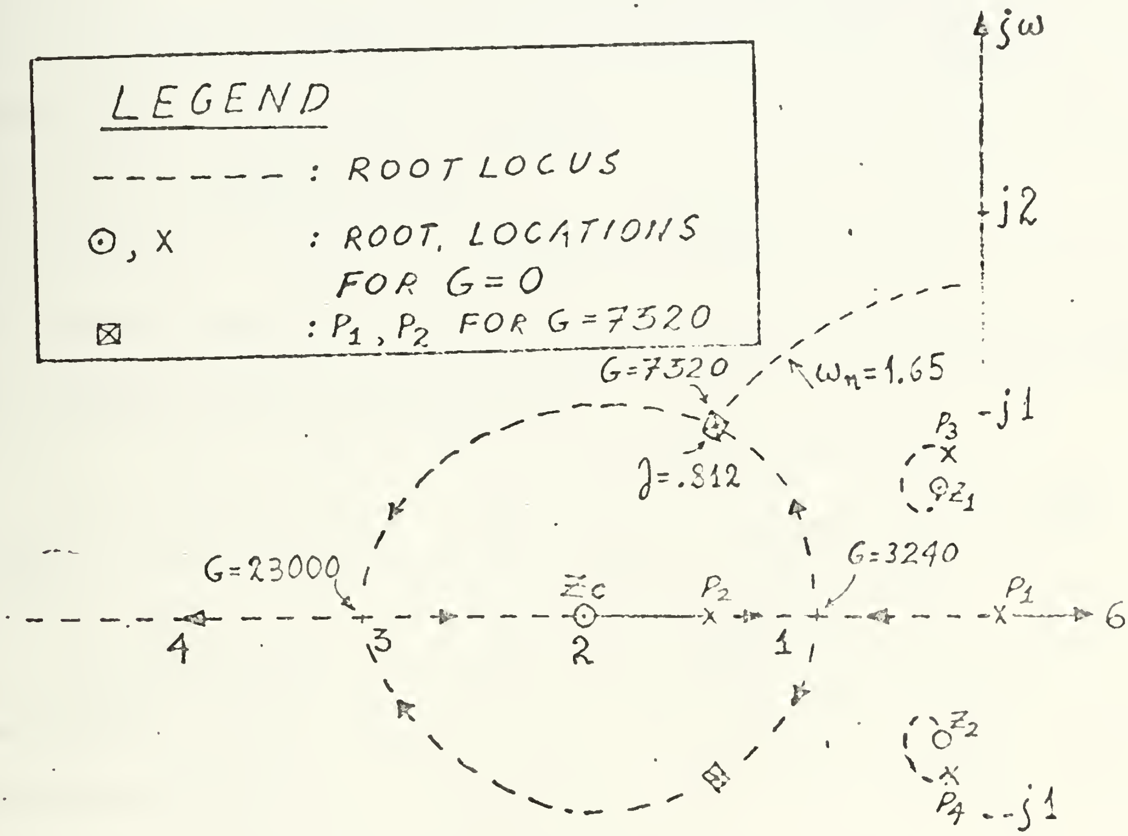
This will result to a real root pair, one of which will be in the right half plane, for "low-enough" gain.

5. After this discussion, it is obvious that two are the possible compensation modes, one with the "z_c" in the L.H.P. and one with "z_c" in the R.H.P. Note that the operating point of the system was regulated for a Left Half Plane compensation with:

$$(K_p, K_\phi) = (95, 190)$$

giving a zero at $z_c = -2$.

ROOT LOCUS FOR
$(K_p, K_\varphi) = (95, 190)$



SCALE
FOR BOTH σ, jω
1 UNIT PER INCH.

FIGURE 6-4a

6. In figure 6-4a, the root locus of the system is plotted for this selection of (K_p, K_ϕ). This was done as usual by equating the open-loop transfer function to minus one, and thus getting:

$$\left[s^6 + 98.5s^5 + 5056.3s^4 + 8966.60s^3 + 6164.65s^2 + 4905.43s - 225.3\right] + G\left[s^3 + 2.443s^2 + 1.371s + .973\right] = 0$$

where G was the variable gain parameter:

$$G = \left[-\frac{(.305) \times K_p \times K}{D}\right]_{(K_p=95)} = 59400\ K,$$

and since the working K of the system was selected to be .1235, one gets the operating point to be as follows:

G	K	K_p	K_ϕ	A	B	ζ	ω_n rad/sec
7320	.1235	95	190	7340	14680	.812	1.65

7. To illustrate by the root-locus technique that compensation is possible for z_c in the Right-Half Plane, one can select $(K_p, K_\phi)=(-95,95)$ and get from the open loop transfer function:

$$\left[s^6 + 98.5s^5 + 5056.3s^4 + 8966.60s^3 + 6164.65s^2 + 4905.43s - 225.3\right] + G'\left[s^3 - .557s^2 + .044s = .487\right] = 0$$

where:

$$G' = \left[\frac{(.305) \times K_p \times K}{D}\right]_{(K_p = -95)} = -59400\ K$$

So for positive K, as in this cass a negative gain locus is resulting and is displayed in figure 6-4b. In this case, K_{max} for stability, is equal to .0965 (equivalent for G' = -5722), which is a restriction for the designer. Notice also that for "ζ" in the area of .707 or greater,

ROOT LOCUS

FOR $(K_P, K_\phi) = (-95, 95)$

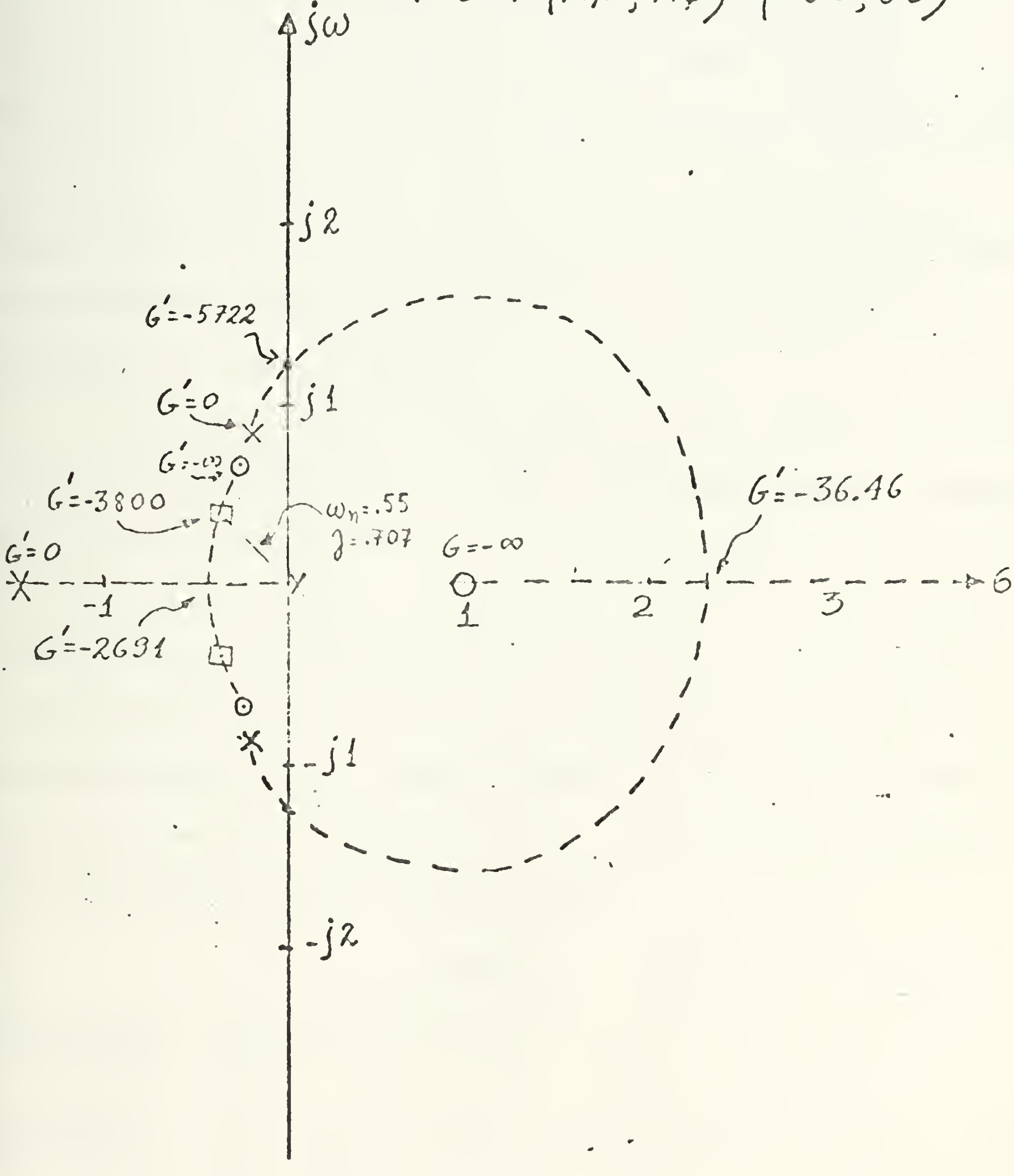


FIGURE 6-4b

the natural frequency will be less than .55rad/sec.

8. With this information about the system, its preliminary investigation is concluded and the parameter plane design follows.

D. DESIGN IN THE PARAMETER PLANE

1. By using the computer program of Appendix D, and the selected sets of constant "ζ" and constant "ω_n" curves shown in table 6-2, three sets of parameter plane grids were plotted for values of the time delay:

$$(T_1, T_2, T_3) = (.15, .2, .3)$$

These results have been plotted in figures 6-5a, 6-5b, and 6-5c. Quasi-linear versions of the maximum root regions found in chapter IV have been superimposed for easier reference.

2. Having in mind the general range of (ζ, ω_n) desired, (section VIB), and the particular $(\zeta, \omega_n) = (.812, 1.65)$ that resulted to be the ones preferably selected by the designers of the system, one can plot the variation of the ($\omega_n = 1.6$ and $\zeta = .8$), constant (ω_n, ζ) curve for varying time delay. This plot and its expanded scale version both resulting from data of table 6-3 are shown in figures 6-6 and 6-6a. Notice the approximately linear relation in these plots, especially within the operating time-delay constants $T \in [.15, .3]$ where ΔT and displacement in the A,B plane remain in a constant ratio.

TABLE 6-2

Constant "ζ" Curve Values	$\zeta_1 = .6$	$\zeta_2 = .65$	$\zeta_3 = .7$	$\zeta_4 = .8$	$\zeta_5 = .85$
Constant "ω" Curve Values	$\omega_1 = .3$	$\omega_2 = .688$	$\omega_3 = 1.0$	$\omega_4 = 1.3$	$\omega_5 = 1.6$

This fact can help in predicting appropriate A,B values for operation with a "time-delay constant" different than $(T_1, T_2, T_3) = (.15, .2, .3)$.

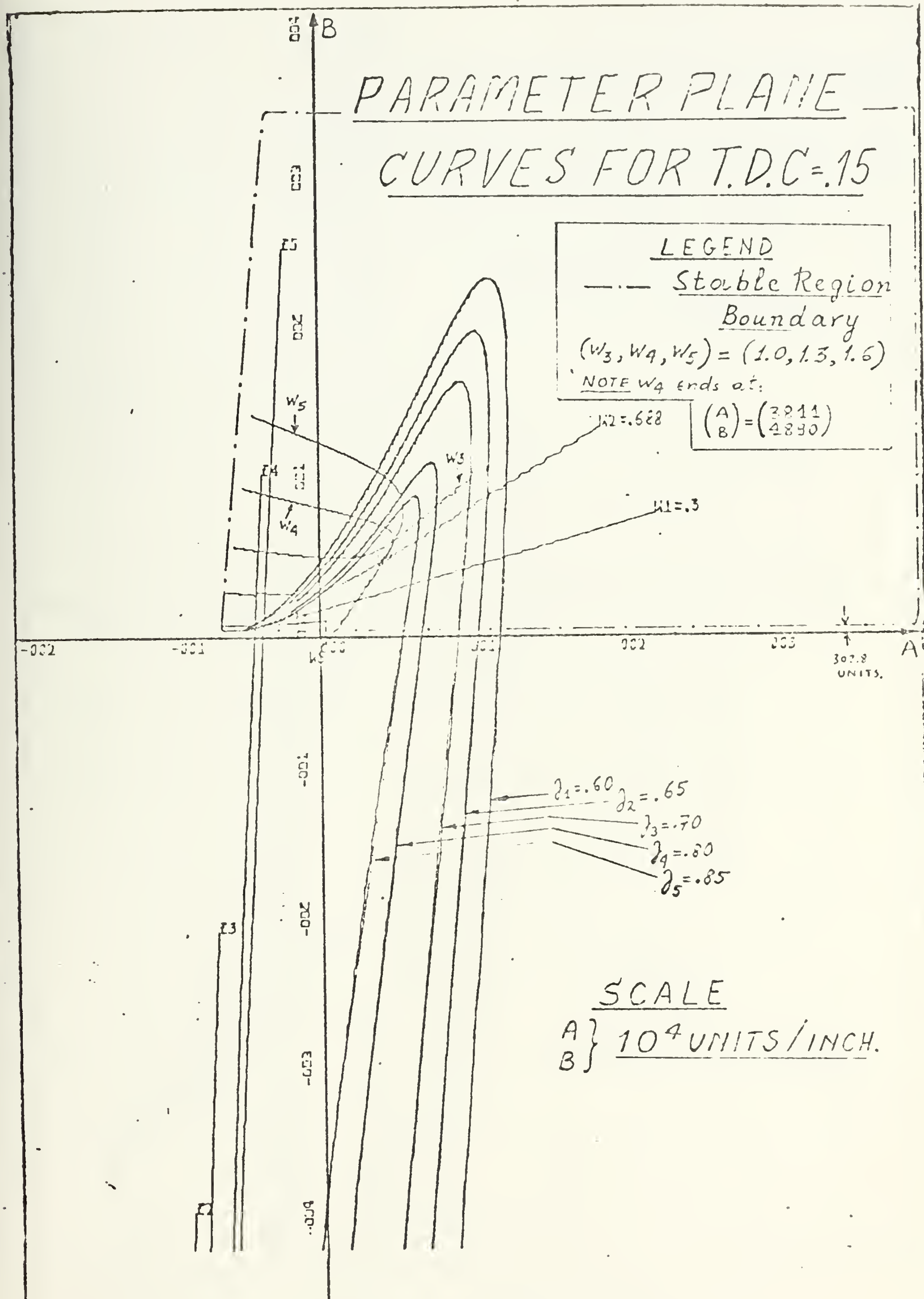

PARAMETER PLANE
CURVES FOR T.D.C=.15
LEGEND
Stable Region Boundary
$(W_3, W_4, W_5) = (1.0, 1.3, 1.6)$
NOTE W_4 ends at:
$\binom{A}{B} = \binom{3211}{4830}$
SCALE
A
B
10^4 UNITS/INCH.
307.8 UNITS.
$\zeta_1=.60$
$\zeta_2=.65$
$\zeta_3=.70$
$\zeta_4=.80$
$\zeta_5=.85$
W1=.3
W2=.688
W_3
W_4
W_5
B
A

FIGURE 6-5a

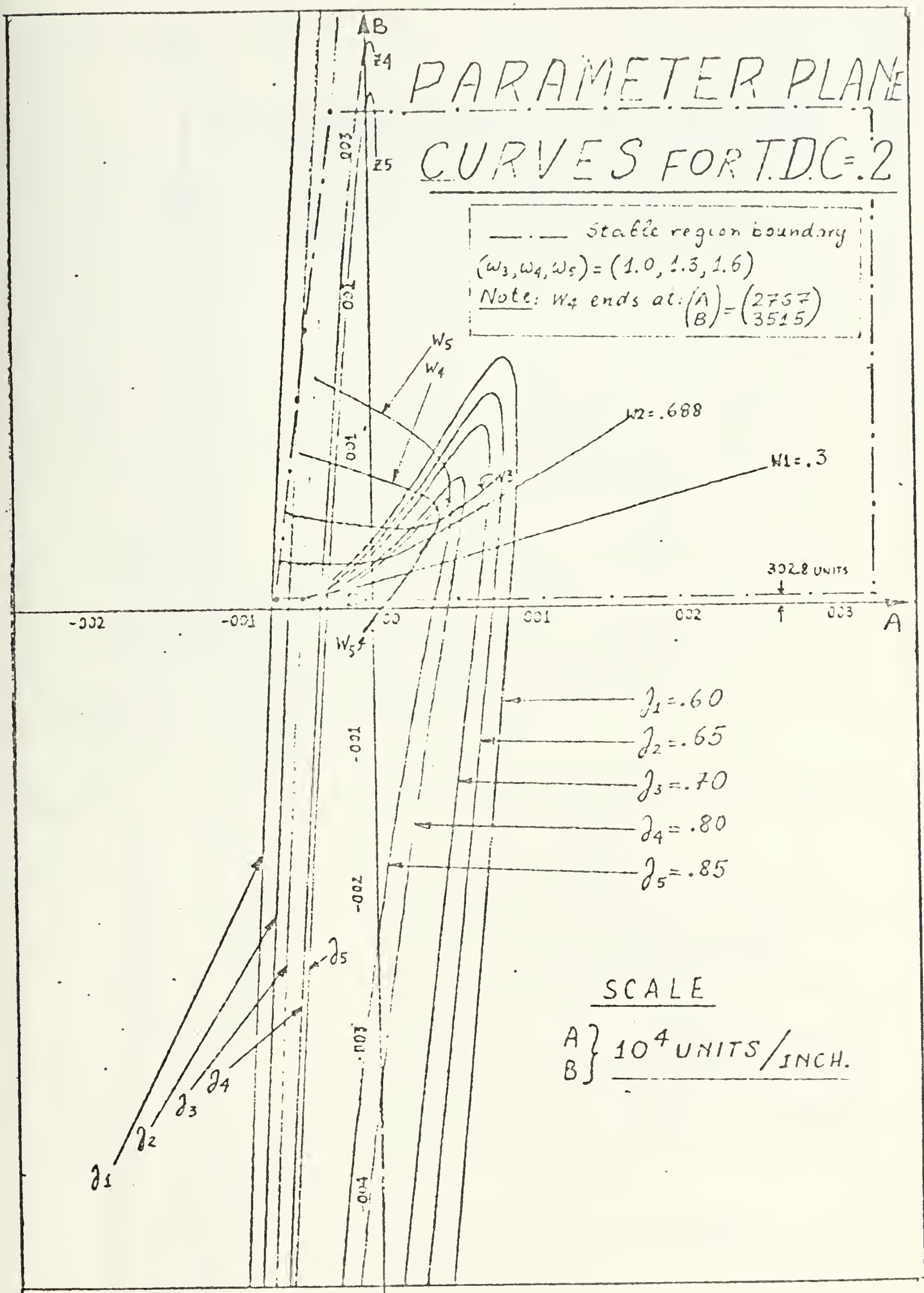

PARAMETER PLANE CURVES FOR T.D.C.=2
Stable region boundary
(ω3, ω4, ω5) = (1.0, 1.3, 1.6)
Note: W4 ends at: (A, B) = (2757, 3515)
W5
W4
W2=.688
W1=.3
302.8 UNITS
A
B
-002
-001
.00
001
002
003
003
.001
-.001
-.002
-.003
-.004
ʒ1=.60
ʒ2=.65
ʒ3=.70
ʒ4=.80
ʒ5=.85
ʒ1
ʒ2
ʒ3
ʒ4
ʒ5
SCALE
A, B } 10⁴ UNITS/INCH.

FIGURE 6-5b

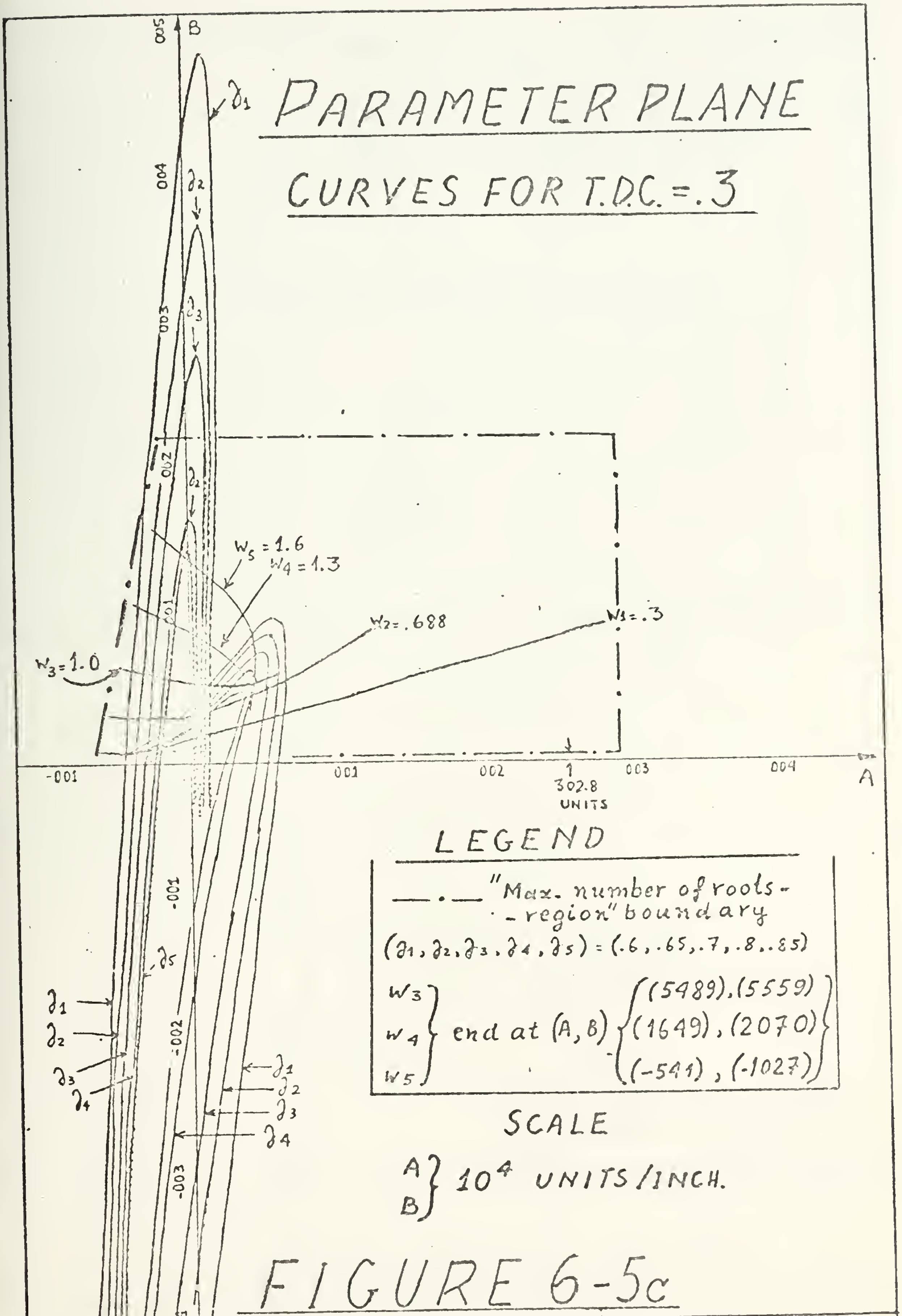

PARAMETER PLANE
CURVES FOR T.D.C. = .3
B
A
.005
.004
.003
.002
.001
-.001
-.002
-.003
-.001
.001
.002
.003
.004
1
302.8
UNITS
$W_5 = 1.6$
$W_4 = 1.3$
$W_2 = .688$
$W_1 = .3$
$W_3 = 1.0$
∂_1
∂_2
∂_3
∂_4
∂_5
LEGEND
"Max. number of roots - region" boundary
$(\partial_1, \partial_2, \partial_3, \partial_4, \partial_5) = (.6, .65, .7, .8, .85)$
W_3, W_4, W_5 end at (A, B) (5489), (5559); (1649), (2070); (-541), (-1027)
SCALE
A, B: 10^4 UNITS/INCH.

FIGURE 6-5c

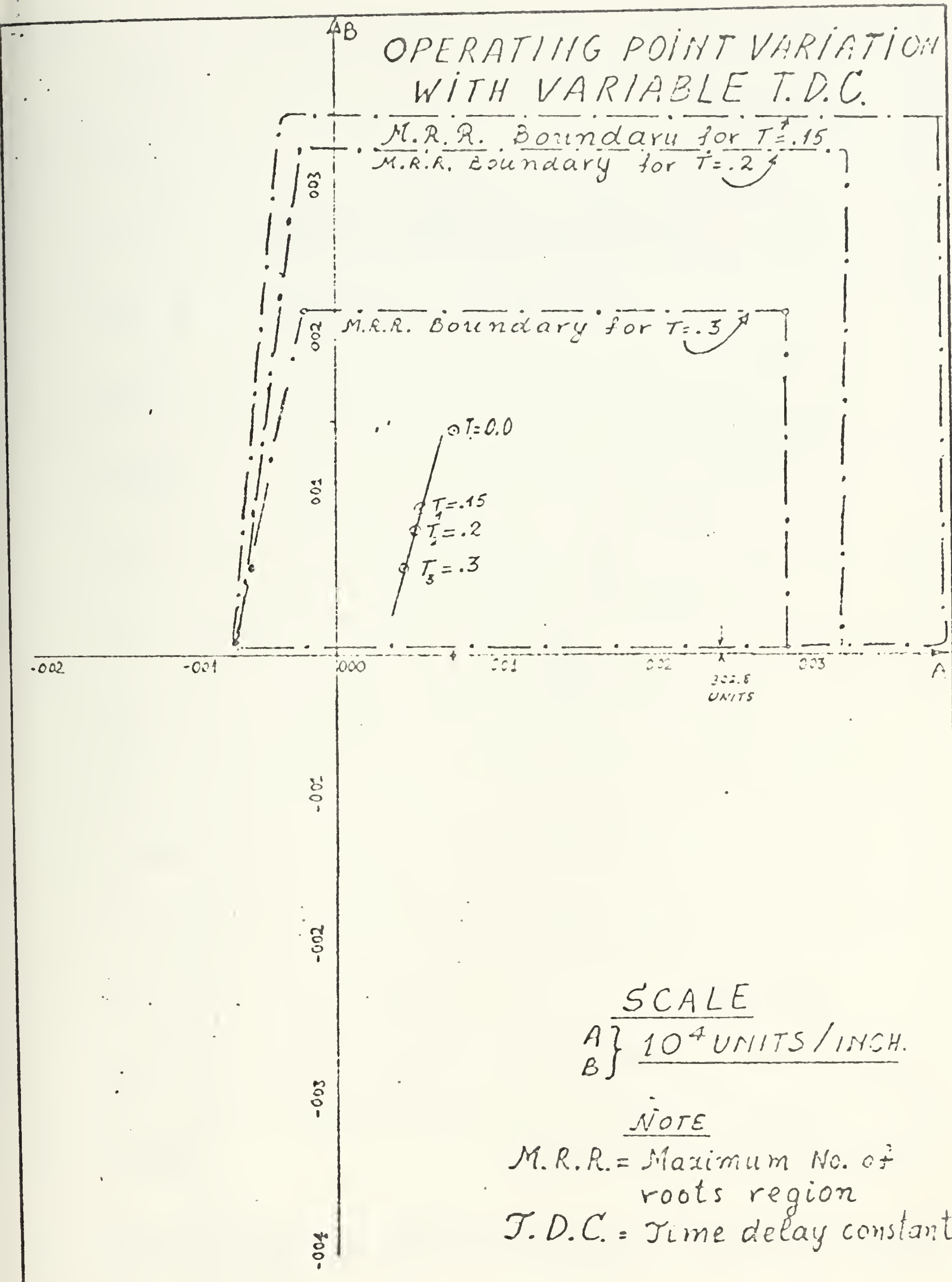

OPERATING POINT VARIATION WITH VARIABLE T.D.C.
M.R.R. Boundary for $T_1 = .15$
M.R.R. Boundary for $T_2 = .2$
M.R.R. Boundary for $T_3 = .3$
$T = 0.0$
$T_1 = .15$
$T_2 = .2$
$T_3 = .3$
A
B
-.002
-.001
.000
.001
.002
.003
-.003
-.004
SCALE
A, B: 10^4 UNITS / INCH.
NOTE
M.R.R. = Maximum No. of roots region
T.D.C. = Time delay constant

FIGURE 6-6

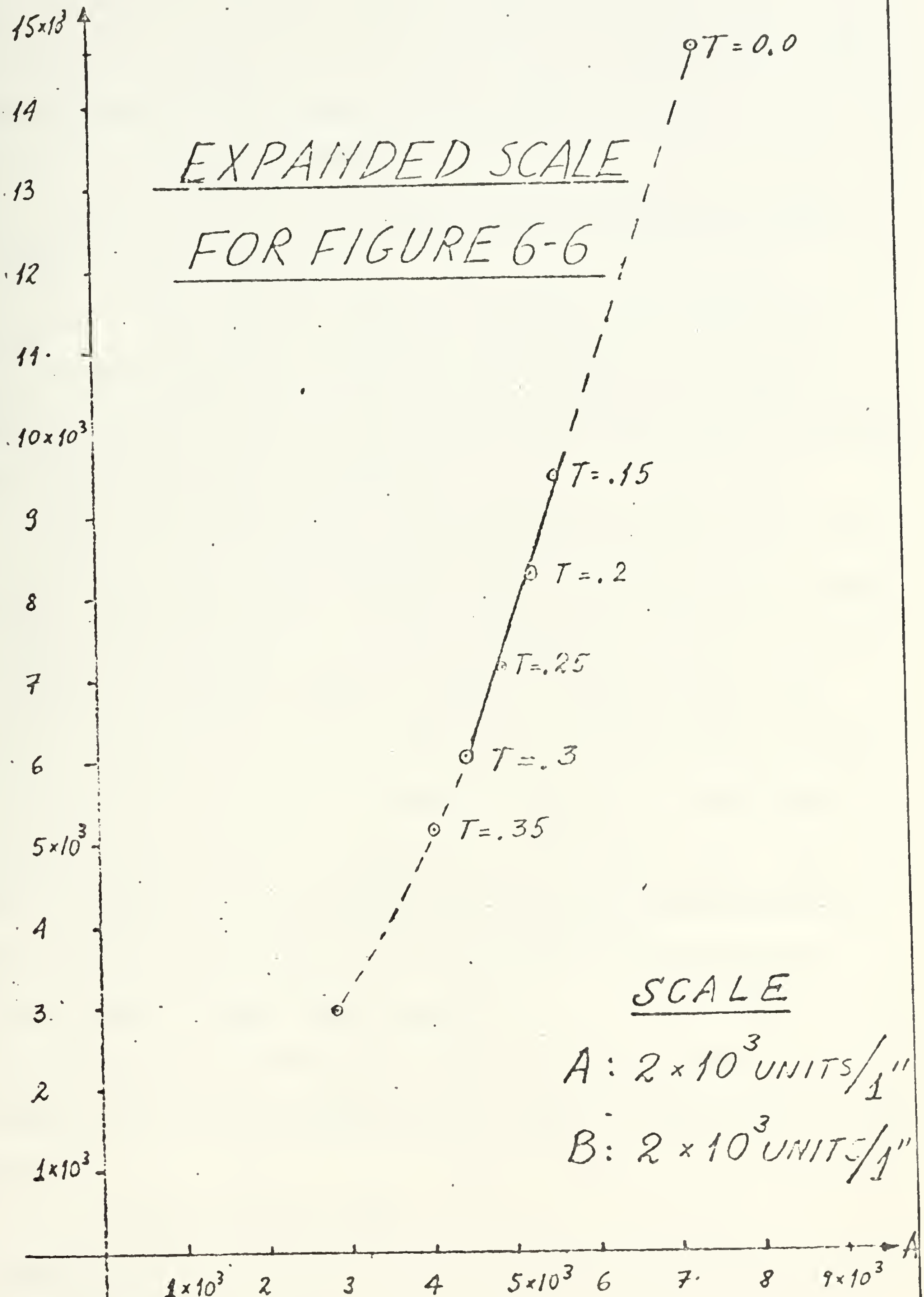

EXPANDED SCALE
FOR FIGURE 6-6
T = 0.0
T = .15
T = .2
T = .25
T = .3
T = .35
SCALE
A : 2 × 10³ UNITS/1"
B: 2 × 10³ UNITS/1"
15×10³
14
13
12
11
10×10³
9
8
7
6
5×10³
4
3
2
1×10³
1×10³
2
3
4
5×10³
6
7
8
9×10³
A

FIGURE 6-6a

However since there exists no limitation on the lower value of the time-delay to be used, or in other words since $T_{min} = .15$ is an admissible delay constant, the best choice of A,B values, in the absence of any other criterion to be met, is the set for T = .15:

$$(A,B) = (5574, 9453)$$

3. After having the first complex pole pair fixed, one should think about the other two pairs.

a. In the case the system is operating within the stable region, one should take into account that the complex pair at $\omega_n = 69$ rad/sec contributes a very small percentage to the total system transient response and therefore can be neglected.

b. If the second pair remains near the $(\omega_n, \zeta) = (.7, .35)$ zero pair, in the case now that finite delay is present, there will again occur small interference on the transient response from P_3, P_4. But in order for this to happen one should be able to observe constant (ω_n) curves of about (.7) all over the stable region for time-delay constant in the range from (.15 ÷ .3). To show that this is true the parameter plane curves were solved for an average time delay value, (T = .225), and plotted in figure 6-7. So by observing this figure one can see the existence of the $\omega_n = 0.7$ curve spreading out across the A,B plane and therefore present for every (A,B) selection.

4. As far as the solution which was obtained by conventional means (root-locus, etc.) one can observe in figure 6-6a that, for T = 0.0, the set

$$(A,B) = (7340, 14680)$$

corresponding to

$$(K_p, K_\phi) = (95, 190),$$

lies in the stable region.

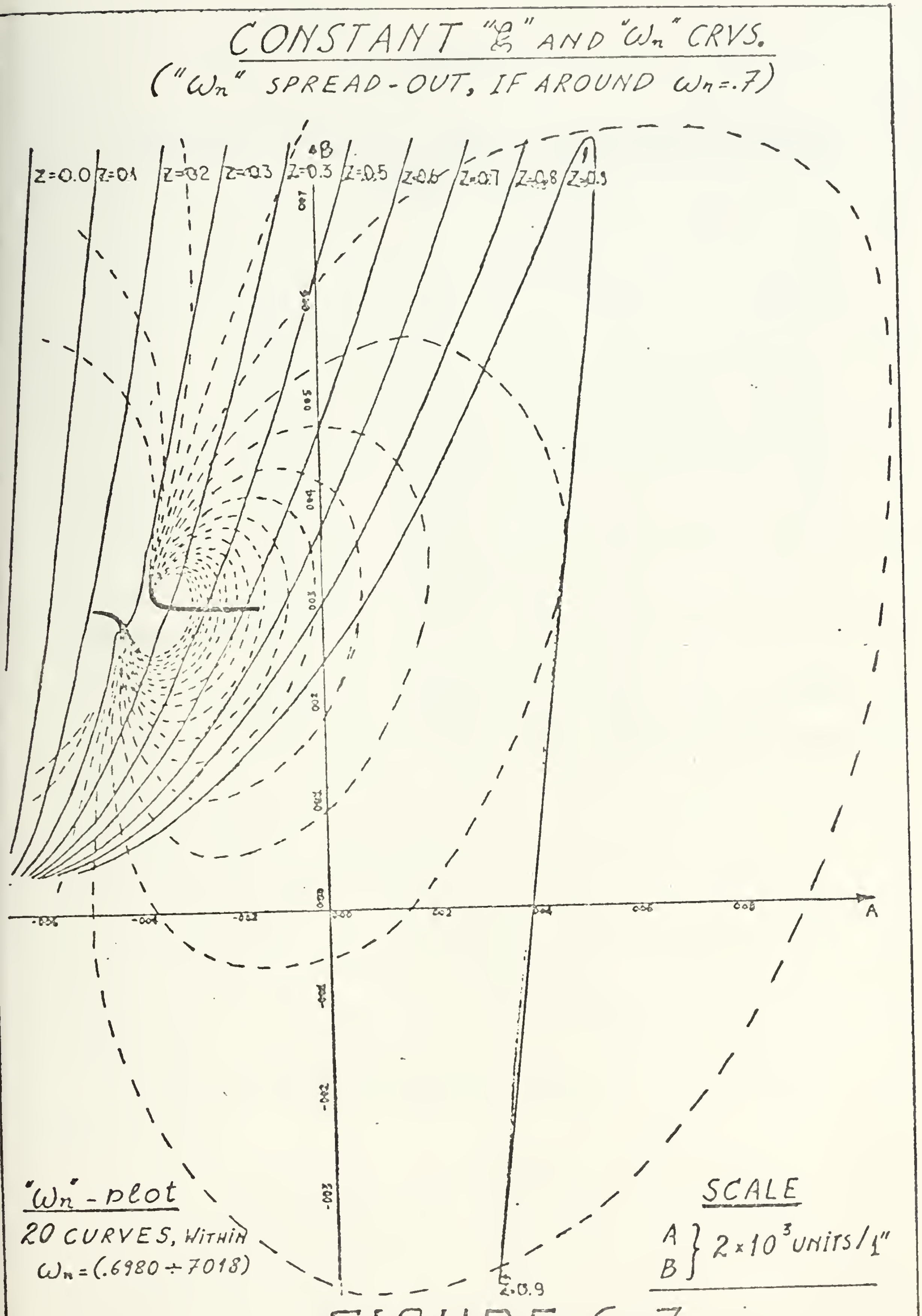

CONSTANT "ζ" AND "ωn" CRVS.
("ωn" SPREAD-OUT, IF AROUND ωn=.7)
B
Z=0.0
Z=0.1
Z=0.2
Z=0.3
Z=0.3
Z=0.5
Z=0.6
Z=0.7
Z=0.8
Z=0.9
A
Z=0.9
"ωn"-plot
20 CURVES, WITHIN
ωn=(.6980 ÷ 7018)
SCALE
A, B } 2×10³ UNITS/1"

FIGURE 6-7

TABLE 6-3

VARIATION OF "A" AND "B" VERSUS "T" FOR CONSTANT OPERATION ON $(\omega_n, \zeta) = (1.6, .8)$

T sec	A units	B units	K_p	K_ϕ
0.0	7340	14680	95	190
0.5	2853	2962	37	38.4
0.1	5722	10568	74.1	136.6
0.15	5574	9453	72.2	122.5
0.2	5280	8279.4	68.4	107.2
0.25	4893	7130	63.3	92.3
0.3	4463.6	6064.1	57.8	78.6
0.35	4026.4	5110.3	52.2	67.5

E. TIME DOMAIN SIMULATION

1. It would be convenient to present at this stage an appropriate transformation of the system from its physical block-diagram to another mathematically equivalent diagram which is appropriate for simulation. Two such mathematical block diagrams are shown in figures 6-2a and 6-2b. The conventions used in figure 6-2a are shown in table 6-4.

2. From this time-domain simulation the time responses of figures 6-3a, to 6-3f were obtained, with the results as shown tabulated in table 6-5.

3. The steady state output one gets from equation (VI-1) is given by the formula

$$\Phi_{ss} = 1 * \frac{36.7}{.488B - 147.59}$$

F. CONCLUSIONS

1. The system proved to be stable for T = .1, .15, .2, .25, .35.

2. The results of section IV and the time simulation coincide as far as the T = .1, T = .2 cases are concerned.

3. For the T = .3 stability analysis, it seems that for both the section's IV method and the time domain simulation, one would expect to get stability for T = .3 by trying bigger frequency and final time ranges respectively. But since it was decided to use T = .15, this was not necessary.

4. The system is safe stability-wise to use for the selected parameters of:

$$A = 5574 \qquad K_p = 72.2$$

$$B = 9453 \qquad K_\phi = 122.5$$

with a time-delay constant of T = .15.

SYSTEMS "T-DOMAIN" SIMULATION MODEL

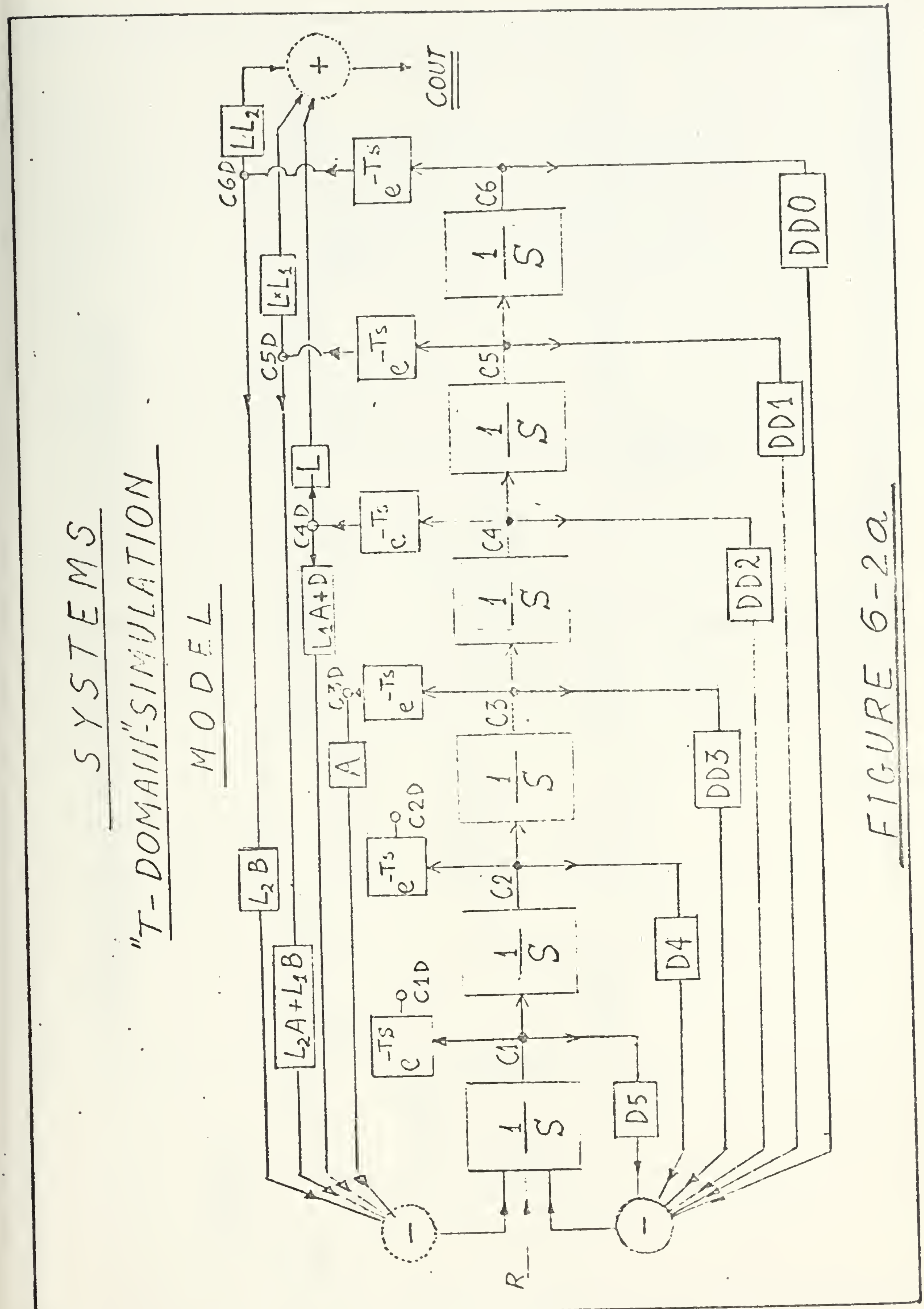


FIGURE 6-2a

MATHEMATICAL BLOCK DIAGRAM, #2

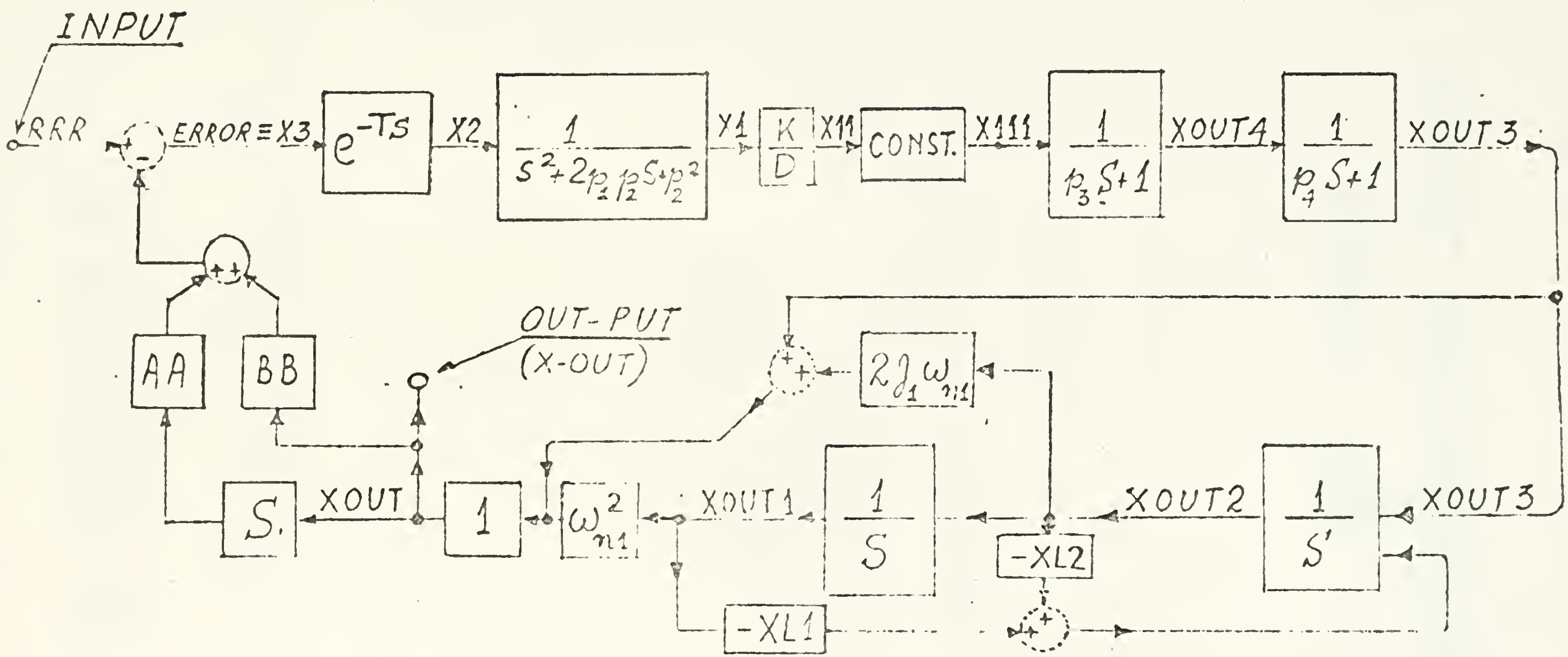
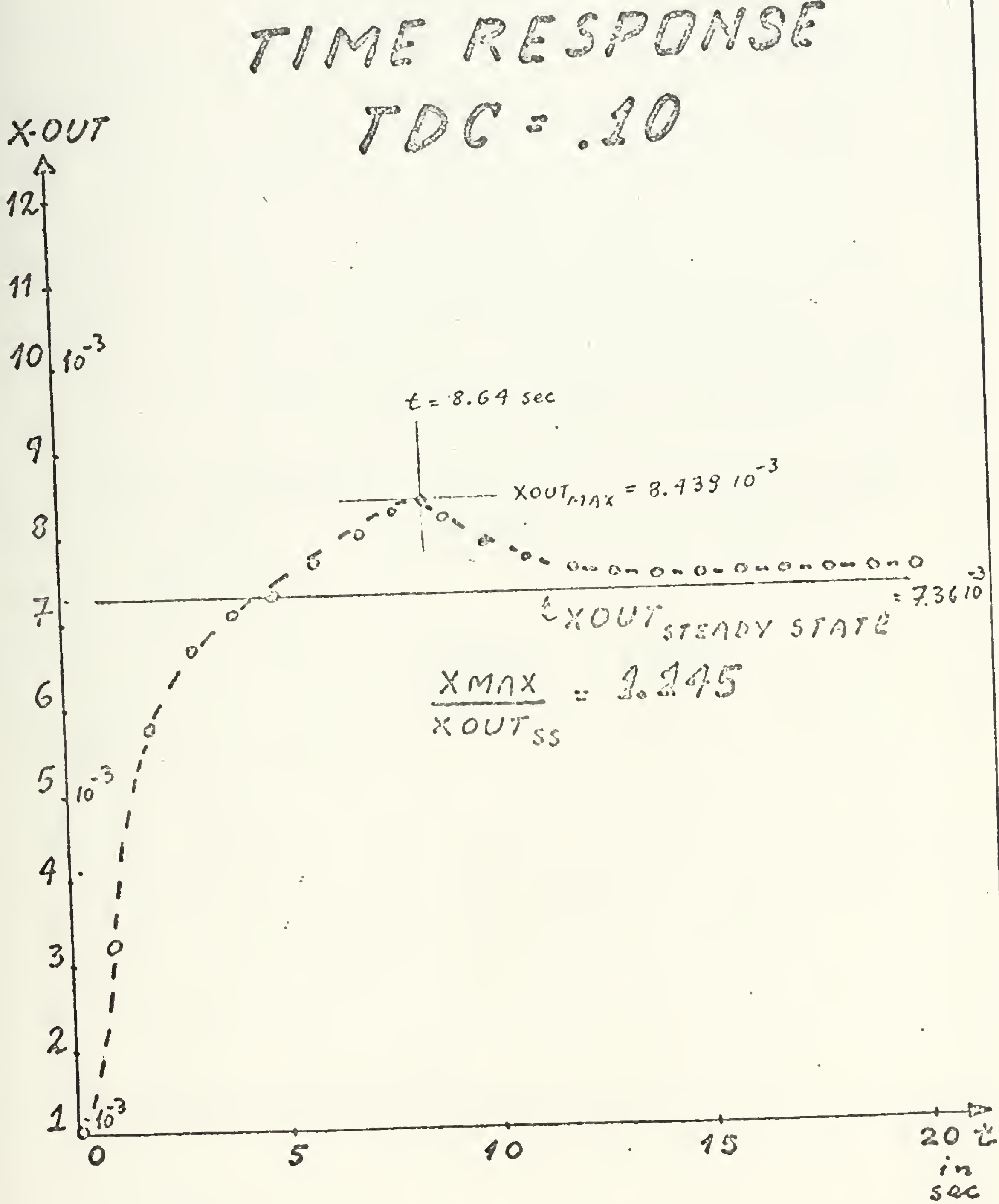


FIGURE 6-26



TIME RESPONSE
TDC = .10
X-OUT
12
11
10 10^-3
9
8
7
6
5 10^-3
4
3
2
1 10^-3
0
5
10
15
20 t in sec
t = 8.64 sec
XOUT MAX = 8.439 10^-3
= 7.36 10^-3
XOUT STEADY STATE
XMAX / XOUT SS = 1.145

FIGURE 6-8A

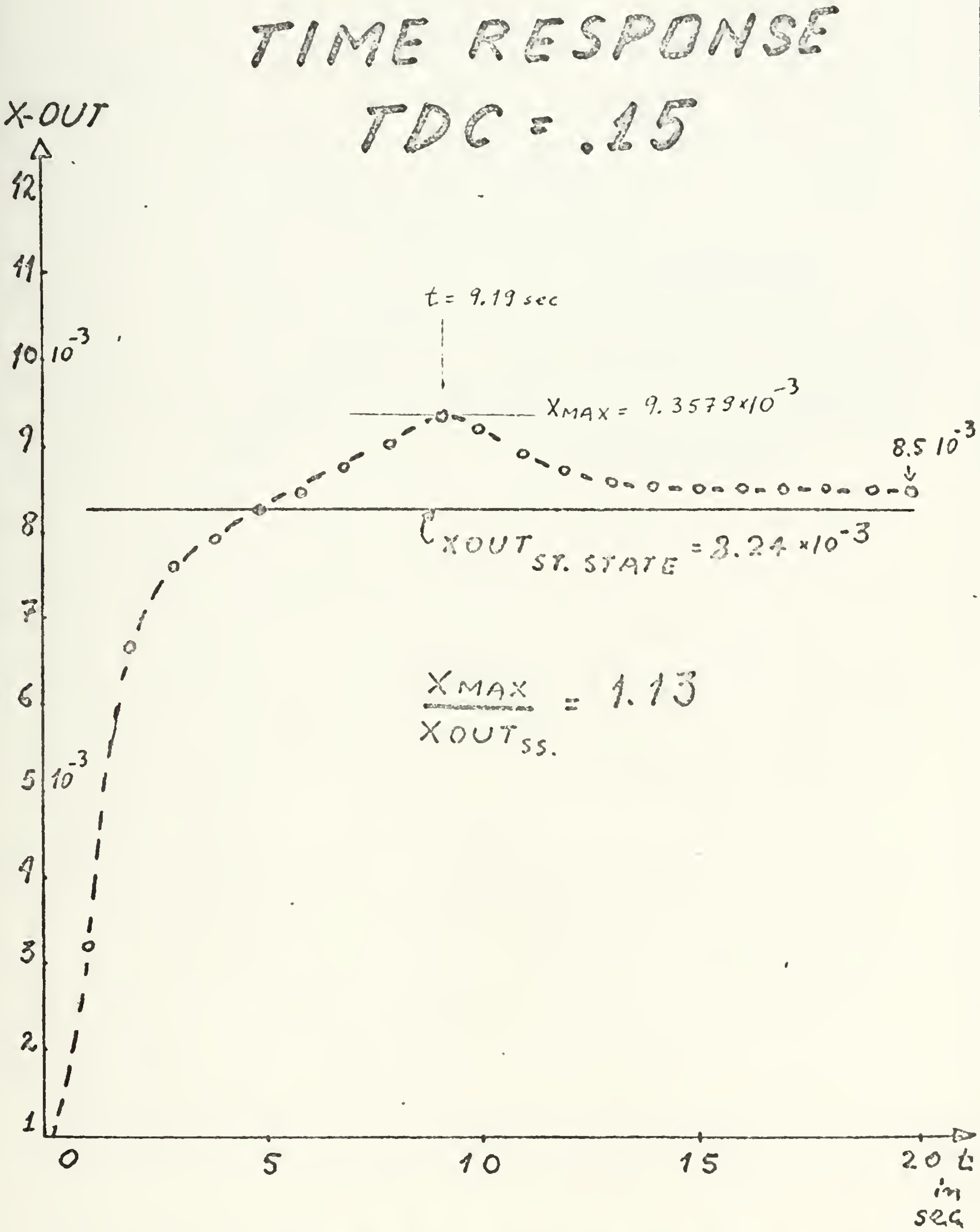

TIME RESPONSE
TDC = .15
X-OUT
12
11
10 10^{-3}
9
8
7
6
5 10^{-3}
4
3
2
1
t = 9.19 sec
$X_{MAX} = 9.3579 \times 10^{-3}$
8.5 10^{-3}
$C_{XOUT_{ST.\ STATE}} = 8.24 \times 10^{-3}$
$\frac{X_{MAX}}{XOUT_{SS.}} = 1.13$
0
5
10
15
20 t in sec

FIGURE 6-8B

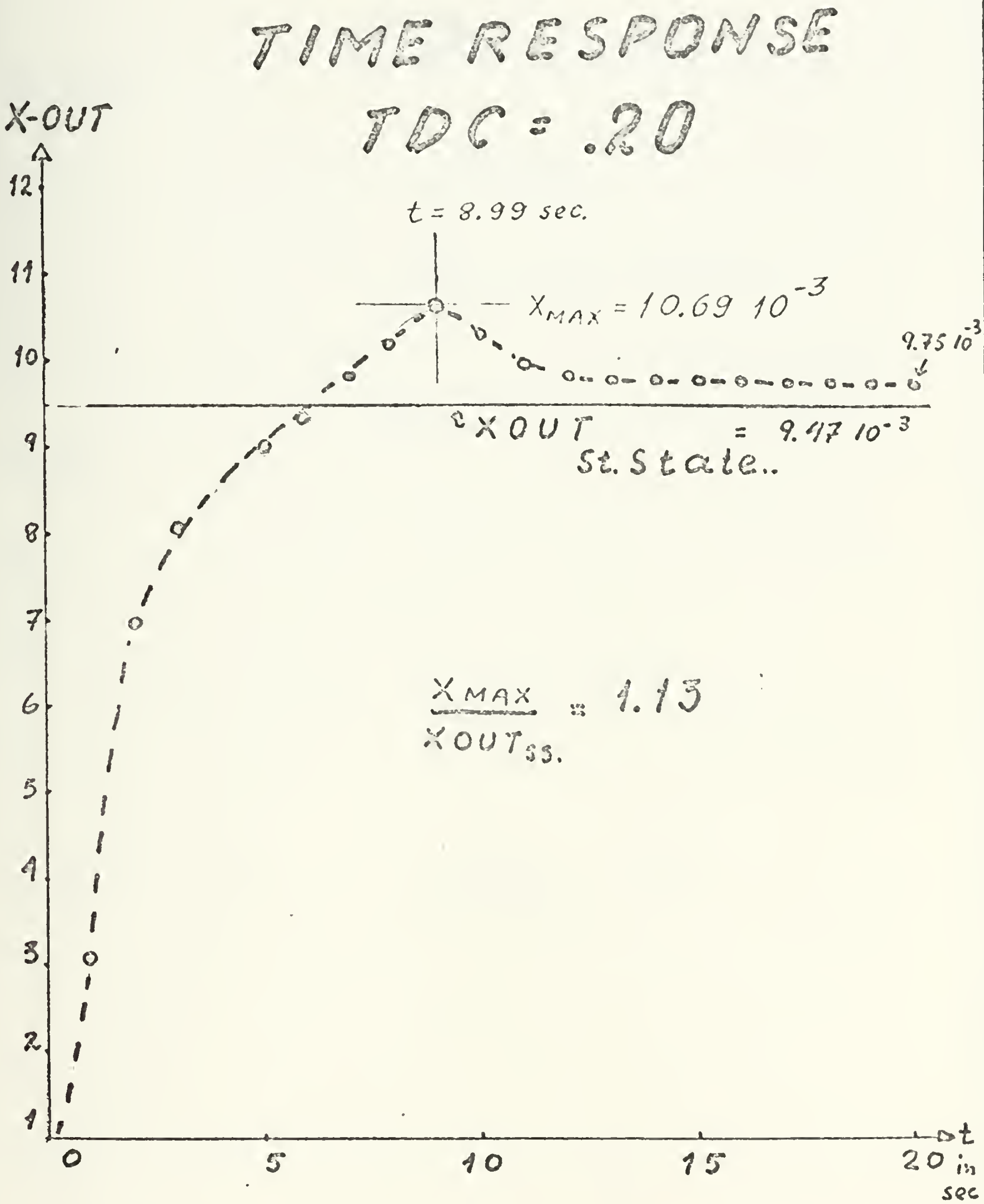

TIME RESPONSE
TDC = .20
X-OUT
12
11
10
9
8
7
6
5
4
3
2
1
t = 8.99 sec.
X MAX = 10.69 10^-3
9.75 10^-3
XOUT St. State.. = 9.47 10^-3
X MAX / XOUT ss. = 1.13
0
5
10
15
20 in sec
t

FIGURE 6-8C

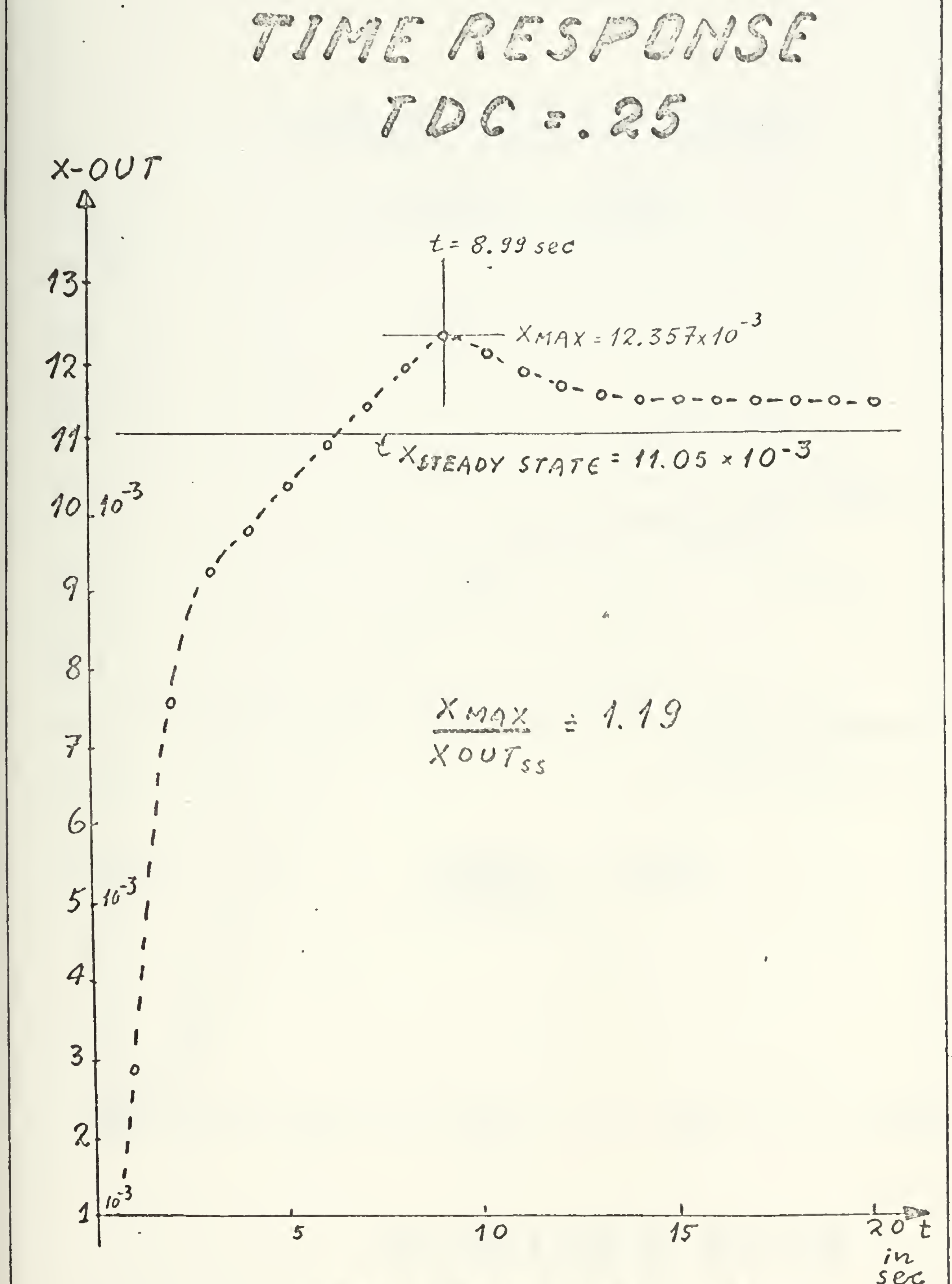

TIME RESPONSE
TDC = .25
X-OUT
t = 8.99 sec
XMAX = 12.357x10-3
XSTEADY STATE = 11.05 x 10-3
XMAX / XOUTss ≐ 1.19
13
12
11
10 10-3
9
8
7
6
5 10-3
4
3
2
1 10-3
5
10
15
20 t
in sec

FIGURE 6-8D

TIME RESPONSE
TDC = .30

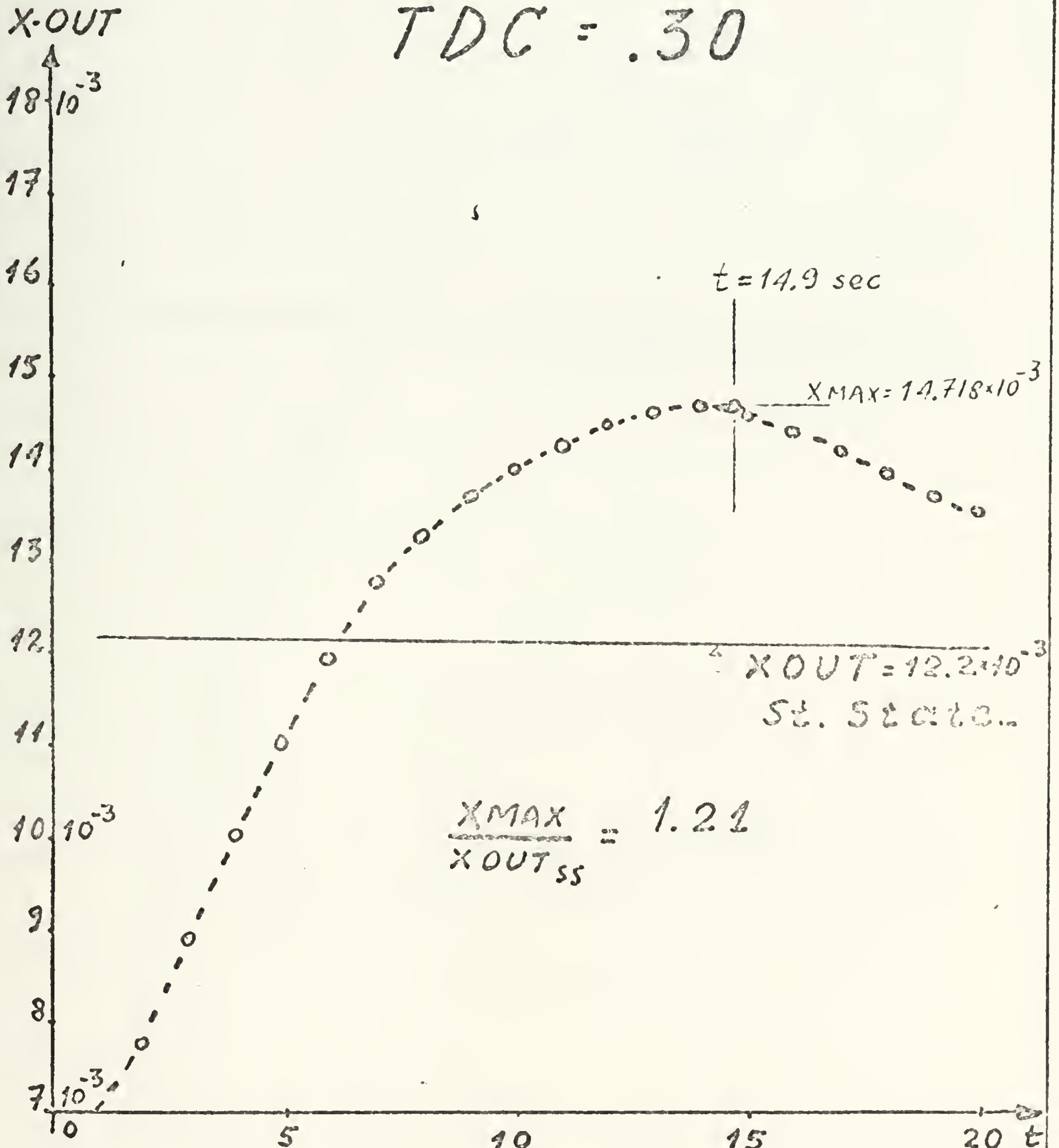


FIGURE 6-8E

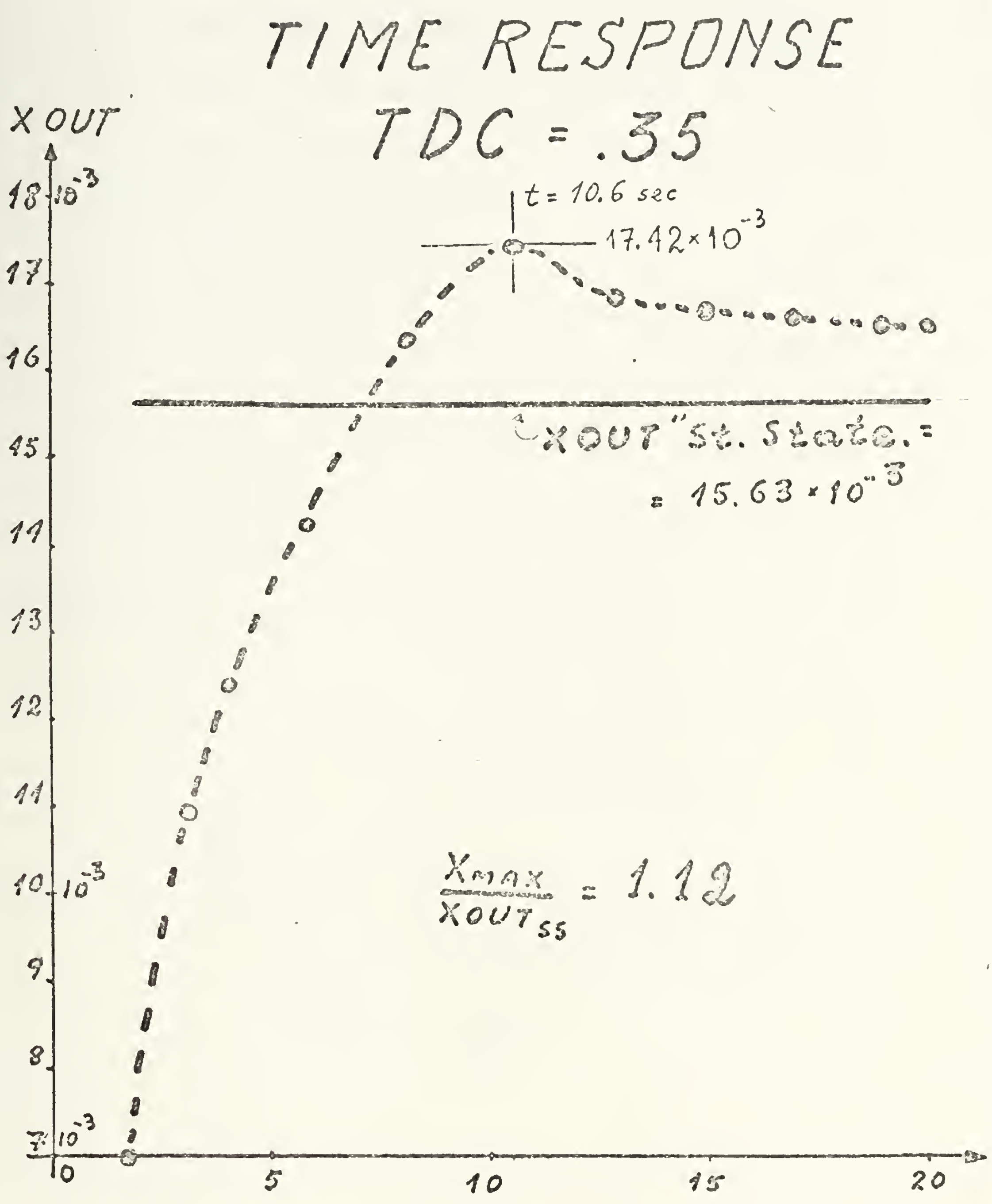

TIME RESPONSE
TDC = .35
X OUT
18 10^-3
17
16
15
14
13
12
11
10 10^-3
9
8
7 10^-3
0
5
10
15
20
t = 10.6 sec
17.42×10^-3
"XOUT" St. State. =
= 15.63×10^-3
Xmax / XOUT_SS = 1.12

FIGURE 6-8F

The time- domain characteristics for a step response are for this case:

α. $\dfrac{\text{"Max. Output"}}{\text{Steady State "Output"}}$ = 1.13

β. Time for "Max Output" = 9.19 sec

γ. Rise Time (10%-90%) = 4.86 sec

δ. Time to reach (1%) error = 10.4 sec

ε. Error at "t=20 sec" = 2.8 ppt

TABLE 6-4

1. Delay

$e^{-Ts} = T_d(s)$

2. Numerator

Call I = 100

$\frac{.305K}{D} = L = 77.2, \ (.444) = L_1, \ (.488) = L_2$

then:

$N_2 = L \times T_d(s)$

$N_1 = L_1 \times L \times T_d(s)$

$N_o = L_2 \times L \times T_d(s)$

3. Denominator

$D_6 = 1.0$

$D_5 = 98.469$

$D_4 = 4941.1680$

$D_3 = 8978.082 + A \times T_d(s)$

$D_2 = 5086.9961 + [(.444) \times A + B] \times T_d(s)$

$D_1 = 3162.6011 + [(.488) \times A + (.444) \times B] \times T_d(s)$

$D_o = -147.59 + [(.488) \times B] \times T_d(s)$

DD3 = 8978.082

DD2 = 5086.9961

DD1 = 3162.6011

DD0 = -147.59

4. With the conventions of table 6-2 at hand, the system's transfer function becomes:

$$\frac{c}{R}(s) = \frac{N_2(s)s^2 + N_1(s)\ s^1 + N_o(s)}{D_6(s)s^6 + D_5(s)s^5 + D_4(s)s^4 + D_3(s)s^3 + D_2(s)s^2 + D_1(s)s + D_o(s)}$$

or

$$\frac{C}{R} = \frac{N_2 \frac{1}{s^4} + N_1 \frac{1}{s^5} + N_o \frac{1}{s^6}}{1 + D_5 \frac{1}{s} + D_4 \frac{1}{s^2} + D_3 \frac{1}{s^3} + D_2 \frac{1}{s^4} + D_1 \frac{1}{s^5} + D_o \frac{1}{s^6}}$$

From this form the signal-flow diagram of figure 6-2a, follows easily.

TABLE 6-5

Time Delay Constant	Maximum Output (10^{-3} units)	Time for Maximum Output (sec)	Maximum Output to Steady State Output (Ratio)	Error at 20 sec (p.p.th)	Remarks
.1	8.439	8.64	1.145	2.72	Stable
.15	9.358	9.19	1.13	2.8	Stable
.2	10.69	8.99	1.13	2.66	Stable
.25	12.36	8.99	1.19	3.26	Stable
.3	14.72	14.9	1.21	10.1	Shows to tend to be stable
.35	17.42	10.6	1.12	5.56	Stable

APPENDIX A

ADDITIONAL INFORMATION CONCERNING THE ROOT-LOCUS ANALYSIS OF A SYSTEM WITH TIME-DELAY

This appendix contains more specific information concerning the Padé approximations as well as data for the characteristic equation of the system:

$$\frac{C}{R} = \frac{N'(s)}{D'(s)} \qquad \text{(see Table II)}$$

resulting from the block diagram of figure 2-2, but with the above mentioned approximation inserted.

<u>The Padé Approximation</u>

If one represents by DE(s), the transfer function of a time-delay in the "s-domain" then for T being the time-delay constant:

$$DE(s) = e^{-Ts} = 1 - Ts + \frac{(Ts)^2}{2!} - \frac{(Ts)^3}{3!} + \dots$$

As already mentioned in section II, the objective of the Padé approximations is to define two polynomials $N_{a,b}(Ts)$ and $D_{a,b}(Ts)$ of order "a" and "b", respectively, which have a quotient

$$F_{a,b}(Ts) = \left[\frac{N_{a,b}(Ts)}{D_{a,b}(Ts)}\right]$$

such that a certain number of terms of both the related series expansion of DE(s) and this quotient are identical.

Letting:

$$F_{a,b}(Ts) = \frac{N_{a,b}(Ts)}{D_{a,b}(Ts)} = \frac{\lambda_o + \lambda_1(Ts) + \lambda_2(Ts)^2 + \dots + \lambda_a(Ts)^a}{\mu_o + \mu_1(Ts) + \mu_2(Ts)^2 + \dots + \mu_b(Ts)^b}$$

one observes that $F_{a,b}(Ts)$, is a function of the order a,b of $N_{a,b}(Ts)$ and $D_{a,b}(Ts)$. So after selecting a,b, one must choose the appropriate values for the coefficients $\lambda_o \div \lambda_a$ and $\mu_o \div \mu_b$ in order to define the Padé approximation.

The polynomials $N_{a,b}(Ts)$ and $D_{a,b}(Ts)$ are specified by:

$$N_{a,b}(Ts) = 1 + \sum_{n=1}^{n=a} (-1)^n \frac{(Ts)^n}{n!} \prod_{m=0}^{m=n-1} \frac{a-m}{a+b-m}$$

$$D_{a,b}(Ts) = 1 + \sum_{n=1}^{n=b} \frac{(Ts)^n}{n!} \prod_{m=0}^{m=n-1} \frac{b-m}{a+b-m}$$

In Table A1, a list of the first six Padé approximations is tabulated.

TABLE A1

LIST OF THE FIRST SIX NORMAL (a=b) AND REDUCED (a=b-1) APPROXIMATIONS

Call <u>Ts = X</u>:

$$F_{1,1}(X) = \frac{2 - X}{2 + X} \qquad F_{0,1}(X) = \frac{1}{1 + X}$$

$$F_{2,2}(X) = \frac{12 - 6X + X^2}{12 + 6X + X^2} \qquad F_{1,2}(X) = \frac{6 - 2X}{6 + 4X + X^2}$$

$$F_{3,3}(X) = \frac{120 - 60X + 12X^2 - X^3}{120 + 60X + 12X^2 + X^3} \qquad F_{2,3}(X) = \frac{60 - 24X + 3X^2}{60 + 36X + 9X^2 + X^3}$$

$$F_{4,4}(X) = \frac{1680 - 840X + 180X^2 - 20X^3 + X^4}{1680 + 840X + 180X^2 + 20X^3 + X^4}$$

$$F_{3,4}(X) = \frac{840 - 360X + 60X^2 - 4X^3}{840 + 480X + 120X^2 + 16X^3 + X^4}$$

$$F_{5,5}(X) = \frac{30240 - 15120X + 3360X^2 - 420X^3 + 30X^4 - X^5}{30240 + 15120X + 3360X^2 + 420X^3 + 30X^4 + X^5}$$

$$F_{4,5}(X) = \frac{15120 - 6720X + 1260X^2 - 120X^3 + 5X^4}{15120 + 8400X + 2100X^2 + 300X^3 + 25X^4 + X^5}$$

$$F_{6,6}(X) = \frac{665280 - 332640X + 75600X^2 - 10080X^3 + 840X^4 - 42X^5 + X^6}{665280 + 332640X + 75600X^2 + 10080X^3 + 840X^4 + 42X^5 + X^6}$$

$$F_{5,6}(X) = \frac{332640 - 151200X + 30240X^2 - 3360X^3 + 210X^4 - 6X^5}{332640 + 181440X + 45360X^2 + 6720X^3 + 630X^4 + 36X^5 + X^6}$$

Note: For more detail on the Padé approximation, see reference [3].

TABLE A2

DATA RELEVANT TO THE CHARACTERISTIC EQUATION OF THE SYSTEM SHOWN IN FIGURE 2-2 OF SECTION II

Characteristic Equation:

$$(0)s^0 + (1680)s^1 + \begin{bmatrix} 840T \\ + \\ 1680 \end{bmatrix} s^2 + \begin{bmatrix} 180T^2 \\ + \\ 840T \end{bmatrix} s^3 + \begin{bmatrix} 20T^3 \\ + \\ 180T^2 \end{bmatrix} s^4 + \begin{bmatrix} T^4 \\ + \\ 20T^3 \end{bmatrix} s^5 + \begin{bmatrix} \\ \\ T^4 \end{bmatrix} s^6 +$$

$$K\left[(1680)s^0 - (840T)s + (180T^2)s^2 - (20T^3)s^3 + (T^4)s^4 \right] = 0$$

Calculations For Repetitive Terms:

Terms	T 0.1	0.2	0.3	1.0
$840T$	84	168	252	840
$180T^2$	1.8	7.2	16.2	180
$20T^3$	.02	.16	.54	20
T^4	10^{-4}	16×10^{-4}	81×10^{-4}	1

APPENDIX B

PARAMETER PLANE (CONSTANT "ω_n" and "ζ") EQUATIONS FOR A SYSTEM WITH A TIME DELAY

The characteristic equation of a system is given to be:

$$\sum_{k=0}^{n} d_K \cdot s^K + \sum_{k=0}^{\ell} a_K'' \, e^{-Ts} \, s^K = 0 \qquad \text{(B-1)}$$

where $\quad a_K'' = Ab_K + Bc_K \qquad$ (B-2)

These are the equations (III-2) and (III-3) which characterize the system of figure 3-1 encountered in chapter 3.

The objective is to derive a set of two equations of the form:

$$A \cdot B_1(\omega_n,\zeta) + B \cdot C_1(\omega_n,\zeta) + D_1(\omega_n,\zeta) = 0$$

$$A \cdot B_2(\omega_n,\zeta) + B \cdot C_2(\omega_n,\zeta) + D_2(\omega_n,\zeta) = 0 \qquad \text{(B-3)}$$

where: $\left[B_i, \; C_i, \; D_i\right]_{i=1,2}$ should be explicit functions of ω_n, and ζ and A,B the unknowns.

Using the fact that

$$s = \omega_n(\cos\theta + j\sin\theta) = \omega_n e^{j\theta}$$

it follows that:

$$\sum_{k=0}^{\ell} a_K'' \cdot e^{-Ts} \cdot s^K = \sum_{k=0}^{\ell} a_K'' \, e^{-T\omega_n(\cos\theta + j\sin\theta)} (\omega_n e^{j\theta})^K =$$

$$\sum_{k=0}^{\ell} a_K'' \, e^{-T\omega_n\cos\theta - jT\omega_n\sin\theta} \, \omega_n^K \, e^{jK\theta} =$$

$$\sum_{k=0}^{\ell} a_K'' \, \omega_n^K \, e^{-T\omega_n\cos\theta} \, e^{j[K\theta - T\omega_n\sin\theta]}$$

Now expanding $e^{j[K\theta - T\omega_n \sin\theta]}$ and getting $e^{-T\omega_n \cos\theta}$ out of the summation operator, one can get:

$$\sum_{k=0}^{\ell} a''_K \, e^{-Ts} \, s^K = \tag{B-4}$$

$$e^{-T\omega_n \cos\theta} \sum_{k=0}^{\ell} a''_K \, \omega_n^{\,K} \left[\cos(K\theta - T\omega_n \sin\theta) + j\sin(K\theta - T\omega_n \sin\theta)\right]$$

Similarly then:

$$\sum_{k=0}^{n} d_K \, s^K = \sum_{k=0}^{n} d_K \omega_n^{\,K} (\cos K\theta + j\sin K\theta) \tag{B-5}$$

By substituting equations (B-4), (B-5) into equation (B-1) and by separating the equation into two parts, one for reals and one for imaginaries, it follows:

$$\sum_{k=0}^{n} d_K \omega_n^{\,K} \cos K\theta + e^{-T\omega_n \cos\theta} \sum_{0}^{\ell} a''_K \, \omega_n^{\,K} \cos(K\theta - T\omega_n \sin\theta) = 0$$

$$\sum_{k=0}^{n} d_K \omega_n^{\,K} \sin K\theta + e^{-T\omega_n \cos\theta} \sum_{0}^{\ell} a''_K \, \omega_n^{\,K} \sin(K\theta - T\omega_n \sin\theta) = 0$$

and by substituting

$a''_K = Ab_K + Bc_K$, (B-2), one arrives to the following two equations:

$$A\left[e^{-T\omega_n \cos\theta} \sum_{k=0}^{\ell} b_K \omega_n^{\,K} \cos(K\theta - T\omega_n \sin\theta)\right] +$$

$$B\left[e^{-T\omega_n \cos\theta} \sum_{k=0}^{\ell} c_K \omega_n^{\,K} \cos(K\theta - T\omega_n \sin\theta)\right] +$$

$$\left[\sum_{k=0}^{n} d_K \, \omega_n^{\,K} \cos K\theta\right] = 0 \tag{B-6a}$$

and:

$$A\left[e^{-T\omega_n \cos\theta}\sum_{k=0}^{\ell} b_K \omega_n^K \sin(K\theta - T\omega_n \sin\theta)\right] +$$

$$B\left[e^{-T\omega_n \cos\theta}\sum_{k=0}^{\ell} c_K \omega_n^K \sin(K\theta - T\omega_n \sin\theta)\right] +$$

$$\left[\sum_{k=0}^{n} d_K \omega_n^K \sin K\theta\right] = 0 \qquad \text{(B-6b)}$$

One should notice that equations (B-6) are more or less of the form of equations (B-3), which are the target, since $\cos\theta$, $\sin\theta$, and θ can be expressed as functions of ζ, so theoretically, the problem is solved. But since in general a digital computer solution is desirable, it is noticed that a summation in K containing

$$\sin(K\theta - T\omega_n \sin\theta) \text{ or } \cos(K\theta - T\omega_n \sin\theta)$$

will not be too easy to evaluate.

So this is the point that one must refer to reference [1] and appreciate the fact that he found some existing recursive relations in his parameter plane analysis without a time-delay. From this fact, it is of value to think that perhaps the same thing applies for the case under consideration

For this reason one can try:

1. <u>First Set of Substitutions</u>.

Call:

$$Y_K = \sin(K\theta - T\omega_n \sin\theta) \qquad \text{(B-7a)}$$

$$X_K = \cos(K\theta - T\omega_n \sin\theta) \qquad \text{(B-7b)}$$

From equation (I-7b), by substituting $\cos\theta = -\zeta$, $\sin\theta = (1-\zeta^2)^{\frac{1}{2}}$, and by appropriate trigonometric expansion it follows that:

$$X_{-1} = -\left(\sqrt{1-\zeta^2}\ \sin\left[T\omega_n\sqrt{1-\zeta^2}\right] + \zeta\ \cos\left[T\omega_n\sqrt{1-\zeta^2}\right]\right)$$

$$X_0 = \cos\left[T\omega_n\sqrt{1-\zeta^2}\right]$$

$$X_1 = \sqrt{1-\zeta^2}\ \sin\left[T\omega_n\sqrt{1-\zeta^2}\right] - \zeta\ \cos\left[T\omega_n\sqrt{1-\zeta^2}\right] \qquad \text{(B-8)}$$

Similar relations can be derived for Y_{-1}, Y_0, Y_1 but this is not necessary.

It can be shown by induction applied to equations (B-7), (see Appendix C), that:

$$X_{K+1} = -2\zeta X_K - X_{K-1} \qquad \text{(B-9a)}$$

$$Y_{K+1} = -2\zeta Y_K - Y_{K-1} \qquad \text{(B-9b)}$$

$$Y_K = \frac{1}{\sqrt{1-\zeta^2}}\left[\zeta X_K + X_{K-1}\right] \qquad \text{(B-9c)}$$

2. <u>Second Set of Substitutions</u>.

Call:

$$U_K = \sin K\theta \qquad \text{(B-10a)}$$

$$T_K = \cos K\theta \qquad \text{(B-10b)}$$

It follows that:

$$T_{-1} = -\zeta$$

$$T_0 = 1$$

$$T_1 = -\zeta \qquad \text{(B-11)}$$

and again by induction from equations (I-10) it can be proved (see Appendix C), that :

$$T_{K+1} = -2\zeta T_K - T_{K-1} \qquad \text{(B-12a)}$$

$$U_{K+1} = -2\zeta U_K - U_{K-1} \qquad \text{(B-12b)}$$

$$U_K = \frac{1}{\sqrt{1-\zeta^2}} \quad (\zeta T_K + T_{K-1}) \tag{B-12c}$$

Now after noting that

$$e^{-T\omega_n \cos\theta} = e^{T\omega_n \zeta}$$

and by using equations (B-7) and (B-10) into equations (B-6), one can get equations of the form (B-3) with:

$$\begin{aligned}
B_1 &= B_1(\omega_n, \zeta, X_K) \\
C_1 &= C_1(\omega_n, \zeta, X_K) \\
D_1 &= D_1(\omega_n, \zeta, T_K) \\
B_2 &= B_2(\omega_n, \zeta, Y_K) \\
C_2 &= C_2(\omega_n, \zeta, Y_K) \\
D_2 &= D_2(\omega_n, \zeta, U_K)
\end{aligned} \tag{B-13}$$

The final step in this derivation is to use equations (B-9c) and (B-12c) which provide Y_K and U_K, in terms (X_K, X_{K-1}) and (T_K, T_{K-1}) respectively.

So finally from (B-6),(B-13), (B-9c), (B-12c) it follows:

$$A\left[e^{T\omega_n\zeta}\sum_{K=0}^{\ell} b_K\omega_n^K X_K(\zeta, \omega_n)\right] + B\left[e^{T\omega_n\zeta}\sum_{K=0}^{\ell} c_k\omega_n^K X_K(\zeta, \omega_n)\right] +$$

$$\left[\sum_{k=0}^{n} d_K\omega_n^K T_K(\zeta)\right] = 0 \tag{B-14a}$$

$$A\left[e^{T\omega_n\zeta}\sum_{K=0}^{\ell} b_k\omega_n^K\left(\zeta\cdot X_K(\zeta,\omega_n) + X_{K-1}(\zeta,\omega_n)\right)\right] +$$

$$B\left[e^{T\omega_n\zeta}\sum_{K=0}^{\ell} c_k\omega_n^K\left(\zeta\cdot X_K(\zeta,\omega_n) + X_{K-1}(\zeta,\omega_n)\right)\right] +$$

$$\left[\sum_{K=0}^{n} d_K\omega_n^K\left(\zeta\cdot T_K(\zeta) + T_{K-1}(\zeta)\right)\right] = 0 \tag{B-14b}$$

where:

$$T_{K+1} = -2\zeta T_K - T_{K-1} \tag{B-12a}$$

$$X_{K+1} = -2\zeta X_K - X_{K-1} \tag{B-9a}$$

Equation (B-8) and (B-11), giving X_{-1}, X_0, X_1 and T_{-1}, T_0, T_1 are also necessary if a computer solution of equations (B-14) is implemented.

The definitions of $[B_i, C_i, D_i]_{i=1,2}$ follow by comparison of equations (B-3) and (B-14):

$$B_1 = e^{T\omega_n\zeta} \sum_{K=0}^{\ell} b_K \omega_n^K X_K(\zeta,\omega_n)$$

$$C_1 = e^{T\omega_n\zeta} \sum_{K=0}^{\ell} c_K \omega_n^K X_K(\zeta,\omega_n)$$

$$D_1 = \sum_{K=0}^{n} d_K \omega_n^K T_K(\zeta)$$

$$B_2 = e^{T\omega_n\zeta} \sum_{K=0}^{\ell} b_k \omega_n^K [\zeta X_K(\zeta,\omega_n) + X_{K-1}(\zeta,\omega_n)]$$

$$C_2 = e^{T\omega_n\zeta} \sum_{K=0}^{\ell} c_K \omega_n^K [\zeta X_K(\zeta,\omega_n) + X_{K-1}(\zeta,\omega_n)]$$

$$D_2 = \sum_{K=0}^{n} d_k \omega_n^K [\zeta T_K(\zeta) + T_{K-1}(\zeta)] \tag{B-15}$$

So if $B_1C_2 - B_2C_1 \neq 0$ one can get from (A-3):

$$A = (C_1D_2 - C_2D_1)/(B_1C_2 - B_2C_1)$$

$$B = (B_2D_1 - D_2B_1)/(B_1C_2 - B_2C_1) \tag{B-16}$$

APPENDIX C

PROOF OF THE VALIDITY OF THE RECURSION RELATIONS USED IN APPENDIX A

1. To prove that the relations (I-9) are true one must first prove their validity for K = 1, and then given the truth of these relations for K = V one must prove their truth for K = V+1.

2. Validity of Relations (B-9) for K = 1.

a. In that case it is required to prove that:

(1) $X_2 = -2\zeta X_1 - X_0$ (C-1)

(2) $Y_2 = -2\zeta Y_1 - Y_0$ (C-2)

(3) $Y_1 = (1-\zeta^2)^{-\frac{1}{2}}[\zeta X_1 + X_0]$ (C-3)

given, of course that,

$$Y_K = \sin(K\theta - T\omega_n \sin\theta)$$

and $$X_K = \cos(K\theta - T\omega_n \sin\theta)$$

b. By direct substitution of X_2, X_1, X_0 and appropriate trigonometric expansions in equation (C-1) one gets:

$$\cos 2\theta \cos(T\omega_n \sin\theta) + \sin 2\theta \sin(T\omega_n \sin\theta) =$$

$$(-2\zeta\cos\theta - 1) \cos(T\omega_n \sin\theta) - 2\zeta\sin\theta \sin(T\omega_n \sin\theta)$$

and taking into account that:

$$\zeta = -\cos\theta$$

it is obvious that:

$$-2\zeta \cos\theta - 1 = 2\cos^2\theta - 1 \equiv \cos 2\theta$$

and $$-2\zeta \sin\theta = 2\sin\theta\cos\theta \equiv \sin 2\theta$$

Therefore, equation (C-1) is true.

c. In a similar way one can get that equation (C-2) is true.

d. Substituting for Y_1, X_1, X_0 in equation (C-3), one gets the following:

$$\sin\theta\ \cos(T\omega_n \sin\theta) - \cos\theta\ \sin(T\omega_n \sin\theta) =$$

$$(1-\zeta^2)^{-\frac{1}{2}}[\zeta\cos\theta\cdot\cos(T\omega_n\sin\theta) + \zeta\sin\theta\cdot\sin(T\omega_n\sin\theta)+\cos(-T\omega_n\sin\theta)]$$

This relation is true since $\zeta = -\cos\theta$ as mentioned before. So equation (C-3) is valid.

3. Now in the case that K=V, one must prove that:

(1) $X_{V+2} = -2\zeta X_{V+1} - X_V$ (C-7)

(2) $Y_{V+2} = -2\zeta Y_{V+1} - Y_V$ (C-8)

(3) $Y_{V+1} = (1-\zeta^2)^{-\frac{1}{2}}[\zeta X_1 + X_0]$ (C-9)

provided that

(4) $X_{V+1} = -2\zeta X_V - X_{V-1}$ (C-4)

(5) $Y_{V+1} = -2\zeta Y_V - Y_{V-1}$ (C-5)

(6) $Y_V = (1-\zeta^2)^{-\frac{1}{2}}[\zeta X_V + X_{V-1}]$ (C-6)

4. The proof for the three sets of equations in pairs of two follows the same pattern, so it is enough to actually prove that the relations (C-4) and (C-7) are true. This can be done in steps as follows:

a. By trigonometric expansion of the relation

$$X_K = \cos\ (K\theta - [T\omega_n \sin\theta])$$

for: $K = V-1,\ V,\ V+1,\ V+2$

$$X_{V-1} = (\cos V\theta\ \cos\theta + \sin V\theta\ \sin\theta)\ \cos[T\omega_n\sin\theta] + (\sin V\theta\ \cos\theta - \sin\theta\ \cos V\theta)\ \sin[T\omega_n\sin\theta]$$

$$X_V = \cos V\theta\ \cos[T\omega_n\sin\theta] + \sin V\theta\ \sin[T\omega_n\sin\theta]$$
$$\equiv (\cos(V+1)\theta\ \cos\theta + \sin(V+1)\theta\ \sin\theta)\cos[T\omega_n\sin\theta] + (\sin(V+1)\theta\ \cos\theta - \cos(V+1)\theta\ \sin\theta)\sin[T\omega_n\sin\theta]$$

$$X_{V+1} = \cos(V+1)\theta \ \cos[T\omega_n \sin\theta] + \sin(V+1)\theta \ \sin[T\omega_n \sin\theta]$$

$$\equiv (\cos V\theta \ \cos\theta - \sin V\theta \ \sin\theta)\cos[T\omega_n \sin\theta] +$$

$$(\sin V\theta \ \cos\theta + \cos V\theta \ \sin\theta)\sin[T\omega_n \sin\theta]$$

$$X_{V+2} = (\cos(V+1)\theta \ \cos[T\omega_n \sin\theta] + \sin(V+1)\theta \ \sin[T\omega_n \sin\theta])\cos\theta +$$

$$(\cos(V+1)\theta \ \sin[T\omega_n \sin\theta] - \sin(V+1)\theta \ \cos[T\omega_n \sin\theta])\sin\theta$$

b. By observing that:

$$X_{V+1} = X_V \cos\theta - \sin\theta \ \sin[V\theta - T\omega_n \sin\theta], \text{ and}$$

$$X_{V-1} = X_V \cos\theta + \sin\theta \ \sin[V\theta - T\omega_n \sin\theta]$$

which added together, they give

$$X_{V+1} + X_{V-1} = 2 \cos\theta \ X_V, \text{ or}$$

$$X_{V+1} = -2\zeta X_V - X_{V-1}$$

$$X_{V+2} = X_{V+L} \cos\theta - \sin\theta \ \sin[(V+1)\theta - T\omega_n \sin\theta] \quad \text{and}$$

$$X_V = X_{V+1} \cos\theta + \sin\theta \ \sin[(V+1)\theta - T\omega_n \sin\theta]$$

which added together give

$$X_{V+2} + X_V = 2 \cos\theta \ X_{V+1}, \text{ or}$$

$$X_{V+2} = -2\zeta X_{V+1} - X_V$$

c. Now since $X_{K+1} = -2\zeta X_K - X_{K-1}$ holds for $K = 1$ as well as for $K = V$, and $V + 1$, it holds for every other value of K. Q.E.D.

5. The proof that equations (I-12) are also valid follows the same procedure discussed from paragraph 1 up to 4 of this Appendix. But in order to prevent unnecessary duplication one can show only some portion of the last part of this proof just for demonstration:

a. Since $T_K = \cos K\theta$ and $U_K = \sin K\theta$ by trigonometric expansion follows:

(1) $T_{K-1} = \cos K\theta \ \cos\theta + \sin K\theta \ \sin\theta$

$T_K = \cos K\theta$

$T_{K+1} = \cos K\theta \ \cos\theta - \sin K\theta \ \sin\theta$

(2) $U_{K-1} = \sin K\theta \cos\theta - \cos K\theta \sin\theta$

$U_K = \sin K\theta$

$U_{K+1} = \sin K\theta \cos\theta + \cos K\theta \sin\theta$

b. By considering the expressions for T_K, T_{K-1}, and U_K, one can get:

$$T_{K-1} = T_K \cos\theta + U_K \sin\theta\text{, or}$$

$$U_K = (\sin\theta)^{-1}(-\cos\theta\, T_K + T_{K-1})\text{, or}$$

$$U_K = (1-\zeta^2)^{-\frac{1}{2}}(\zeta T_K + T_{K-1}) \qquad \text{(C-10)}$$

c. One can observe also that:

$$T_{K-1} = \cos\theta\left(\cos K\theta + \frac{\sin\theta}{\cos\theta} \sin K\theta\right)$$

$$T_{K+1} = \cos\theta\left(\cos K\theta - \frac{\sin\theta}{\cos\theta} \sin K\theta\right)$$

Therefore, by adding these two equations, the recurrence relation needed, follows:

$$T_{K+1} + T_{K-1} = 2 \cos K\theta = 2T_K\text{, or}$$

$$T_{K+1} = -2\zeta T_K - T_{K-1} \qquad \text{(C-11)}$$

$$U_{K-1} = \cos\theta\left(U_K - \frac{\sin\theta}{\cos\theta} \cos K\theta\right)$$

$$U_{K+1} = \cos\theta\left(U_K + \frac{\sin\theta}{\cos\theta} \cos K\theta\right)$$

and again by addition it follows:

$$U_{K+1} = -2\zeta U_K - U_{K-1} \qquad \text{(C-12)}$$

APPENDIX D

COMPUTER PROGRAM TO IMPLEMENT
SOLUTIONS OF THE LINEAR-FEEDBACK
CONTROL SYSTEM DISCUSSED IN SECTION III A

```
C     PROGRAM PARAM A/DELAY ACCOUNTING FOR T.DELAY ACTION.
C     THE POWER UP TO WHICH THE TDELAY IS SHOWN IN THE CHARACTERISTIC
C     EQUATION IS DENOTED BY    IPOWER (OR    LL    SOMETIMES).
C     THE TIME-DELAY CONSTANT IS DENOTED BY    TDC   .
C     THIS PROGRAM IS APPLICABLE TO POLYNOMIALS WHOSE COEFFICIENTS ARE OF THE
C     FORM (EXP(-T*S)*(B*ALFA+C*BETA)+D) WHERE ALFA AND BETA ARE VARIABLE PARAMS.
C     AND B. C. AND D ARE CONSTANTS.  THIRD PARAMETERS CAN ALSO DE SPECIFIED
C     AS INDICATED BELOW.
C     THIS PROGRAM WILL PLOT ON ONE 9 INCH BY 15 INCH GRAPH,PARAMETER PLANE
C     CURVES OF THE FOLLOWING TYPE. CONSTANT ZETA CURVES AS A FUNCTION OF OMEGA
C     (THE STARTING VALUE OF OMEGA AND THE NUMBER OF DECADES THAT OMEGA WILL
C     SPAN WILL BE SPECIFIED IN THE DATA CARDS). CONSTANT OMEGA CURVES FOR
C     PRE'PROGRAMMED VALUES OF ZETA BETWEEN ZERO AND ONE. CONSTANT SIGMA LINES,
C     THE VALUES OF ZETA FOR THE CONSTANT ZETA
C     CURVES. THE VALUES OF OMEGA FOR THE CONSTANT OMEGA CURVES, THE VALUES OF
C     SIGMA FOR THE CONSTANT SIGMA LINES, AND THE VALUES OF ZETA'OMEGA FOR THE
C     CONSTANT ZETA'OMEGA CURVES MAY BE SPECIFIED IN THE DATA CARDS.
C     IF HOWEVER NO CURVES OF A CERTAIN TYPE ARE DESIRED PLACE A ZERO
C     IN THE APPROPRIATE COLUMN CORRESPONDING TO THE NUMBER OF CURVES.IN
C     THIS CASE SUBMIT A BLANK CARD FOR THE LABELS AND FOR THE CURVE VALUES
C     IF NO CONSTANT ZETA CURVES ARE DESIRED, SET NZ AND ND TO ZERO, AND
C     SUBMIT A BLANK CARD FOR THE ZETA LABELS, FOR THE ZETA CURVE VALUES, AND
C     FOR THE STARTING VALUE OF OMEGA.
C     ALL CURVES ARE PLOTTED ON THE SAME GRAPH.
C     AN ADDITIONAL FEATURE OF THE PROGRAM IS THAT FAMILIES OF CONSTANT
C     ZETA,OMEGA, SIGMA, AND ZETA-OMEGA CURVES MAY BE PLOTTED IN TERMS OF A
C     THIRD PARAMETER. UP TO 10 VALUES OF THE THIRD PARAMETER MAY BE SPECIFIED
C     THE THIRD PARAMETER MAY APPEAR LINEARILY OR NON-LINEARILY IN ANY OF THE
C     COEFFICIENTS. THE X-AXIS VARIABLE IS ALPHA AND THE Y-AXIS VARIABLE IS BETA
C     CONSTANT SIGMA LINES WILL BE COMPUTED ONLY FOR THOSE VALUES OF SIGMA ALONG
C     THE NEGATIVE REAL AXIS IN THE S-PLANE. THESE SIGMA VALUES SHOULD BE
C     ENTERED IN THE DATA CARDS AS POSITIVE QUANTITIES HOWEVER
C     THE FOLLOWING SYMBOLS ARE PERTINANT TO THE PROGRAM,
C     ND- THE NUMBER OF DECADES SPANNED BY OMEGA FOR THE CONSTANT ZETA CURVES
C     NO-THE ORDER OF THE EQUATION, NZ,NS,NW, AND NZW -THE NUMBER OF CONSTANT
C     ZETA. SIGMA. OMEGA, AND ZETA-OMEGA CURVES RESPECTIVELY,NE-THE NUMBER OF
C     VALUES OF THE THIRD PARAMETER. IXUP-DISTANCE IN INCHES OF THE X-AXIS FROM
C     THE BOTTOM OF THE GRAPH, IYRIGHT- THE DISTANCE IN INCHES OF THE Y-AXIS
C     FROM THE LEFT SIDE OF THE GRAPH, LABZ,LABS,LABW,LABZW-THE LABELS FOR THE
C     CONSTANT ZETA,SIGMA, OMEGA, AND ZETA-OMEGA CURVES, WN- THE STARTING VALUE
```

```
C     OF OMEGA FOR THE CONSTANT ZETA CURVES, E-THE THIRD PARAMETER, B5,C5,D5
C     ALPHA,BETA, AND CONSTANT COEFFICIENTS RESPECTIVELY.
C     IF A THIRD PARAMETER IS NOT SPECIFIED THE DATA CARDS ARE SUBMITTED IN THE
C     FOLLOWING MANNER.
C     CARD 1 THE FIRST LINE OF THE GRAPH TITLE (IN COLUMNS 1-48)
C     CARD 2 THE SECOND LINE OF THE GRAPH TITLE (IN COLUMNS 1-48)
C     CARD 3 IN 8I10 FORMAT ENTER FROM LEFT TO RIGHT
C       ND          NO          NZ          NS          NW          NZW       IXUP    IYRIGHT
C     CARD 4 IN COLUMN 10 ENTER A 1 IF PRINTOUT IS DESIRED. LEAVE BLANK IF NO
C     PRINTOUT IS DESIRED.
C     CARD 5 LABZ(20A4 FORMAT), LEAVE BLANK IF NZ=0
C     CARD 6 LABS(20A4 FORMAT), LEAVE BLANK CARD IF NS=0
C     CARD 7 LABW(20A4 FORMAT), LEAVE BLANK IF NW=0
C     CARD 8 LABZW(20A4 FORMAT),LEAVE BLANK CARD IF NZW=0
C     CARD 9 VALUES OF ZETA FOR CONSTANT ZETA CURVES. (8F10.5 FORMAT)
C     CARD 10 VALUES OF SIGMA FOR CONSTANT SIGMA CURVES(8F10.5 FORMAT)
C     CARD 11 VALUES OF OMEGA  FOR CONSTANT OMEGA CURVES(8F10.5 FORMAT)
C     CARD 12 VALUES OF ZETA-OMEGA FOR CONSTANT ZETA-OMEGA CURVES(8F 10.5 FORMAT)
C     CARD 13 CONSTANT COEFFICIENTS IN ASCENDING ORDER (8F10.5 FORMAT)
C     CARD 14 ALPHA COEFFICIENTS IN ASCENDING ORDER (8F10.5 FORMAT)
C     CARD 15 BETA COEFFICIENTS IN ASCENDING ORDER (8F10.5 FORMAT)
C     CARD 16 OMIT
C     CARD 17 WN (E10.5 FORMAT)
C     CARD 18 XSCALE (E10.5 FORMAT, USE 1 SIGNIFICANT FIGURE)
C     CARD 19 YSCALE (E10.5 FORMAT, USE 1 SIGNIFICANT FIGURE)
C     CARD 20 IN 2I10 FORMAT ENTER :
C      IPOWER      SCRAP
C     CARD 21 TDC (E10.5 FORMAT,USE 1 SIGNIFICANT FIGURE)
C         IF A THIRD PARAMETER IS SPECIFIED THE DATA CARDS ARE SUBMITTED IN
C     THE FOLLOWING MANNER.
C     CARD 1 SAME AS CARD 1 IN PREVIOUS SECTION
C     CARD 2 SAME AS CARD 2 IN PREVIOUS SECTION
C     CARD 3 SAME AS CARD 3 IN PREVIOUS SECTION
C     CARD 4 SAME. EXCEPT ENTER THE VALUE OF NE IN COLUMNS 11' 20(USE I FORMAT)
C     CARD 5 SUBMIT NE GROUPS OF LABELS WITH NZ   LABELS IN EACH GROUP. SUBMIT
C     IN CONSECUTIVE ORDER. (20A4 FORMAT), SUBMIT BLANK CARD IF NZ=0.
C     CARD 6 SAME AS CARD 5 ONLY NS IN EACH GROUP, SUBMIT BLANK CARD IF NS=0
C     CARD 7 SAME AS CARD 5 ONLY NW IN EACH GROUP,SUBMIT BLANK CARD IF NW=0
C     CARD 8 SAME AS CARD 5 ONLY NZW IN EACH GROUP, SUBMIT BLANK CARD IF NZW=0
C     CARD 9 SAME AS CARD 9 IN PREVIOUS SECTION
C     CARD 10 SAME AS CARD 10 IN THE PREVIOUS SECTION
C     CARD 11 SAME AS CARD 11 IN THE PREVIOUS SECTION
C     CARD 12 SAME AS CARD 12 IN THE PREVIOUS SECTION
C     CARD 13 OMIT
C     CARD 14 OMIT
C     CARD 15 OMIT
C     CARD 16 VALUES OF THE THIRD PARAMETER (8E10.5 FORMAT)
C     CARD 17 SAME AS CARD 17 IN PREVIOUS SECTION
C     CARD 18 SAME AS CARD 18 IN PREVIOUS SECTION
```

```
C     CARD 19 SAME AS CARD 19 IN PREVIOUS SECTION
C     CARD 20 SAME AS CARD 20 IN PREVIOUS SECTION
C     CARD 21 SAME AS CARD 21 IN PREVICUS SECTION
C           THE COEFFICIENTS OF THE CHARACTERISTIC EQUATION  MUST BE ENTERED
C     SUBROUTINE COEF. FOR EXAMPLE, GIVEN AN EQUATION OF THE FOLLOWING FORM
C     WHERE E CORRESPONDS TO THE THIRD PARAMETER,    DJ(5)*S**4 +
C     (BJ(4)*ALPHA + CJ(4)*BETA + DJ(4))S**3 + (BJ(3)*ALPHA + CJ(3) * BETA +
C     DJ(3))S**2 + (BJ(2)*ALPHA + CJ(2)*BETA + DJ(2))S + (BJ(1) *ALPHA +
C     CJ(1)*BETA + DJ(1))=0
C           THE COEFFICIENTS OF THE ABOVE EQUATION WOULD BE ENTERED EACH ON A
C     SEPARATE CARD BETWEEN THE COMMON AND RETURN CARDS IN SUB ROUTINE COEFF.
C           FOR EXAMPLE
C     COMMON E,BJ,CJ,DJ
C     BJ(1) =2.*E
C     CJ(1) =1.
C     DJ(1) =3.
C     BJ(2)=10.*E
C     CJ(2)=286.+28.6*E
C     DJ(2)=0.
C     BJ(3)=11.
C     CJ(3)=E
C     DJ(3)=10.
C     BJ(4)=1.
C     CJ(4)=0.
C     DJ(4)=11.
C     BJ(5)=0.
C     CJ(5)=0.
C     DJ(5)=1.
C     RETURN
      REAL*4 DUM/1H1/,LABEL/4H    /
      REAL*8ITITLF(12)
      DIMENSION A(350),B(350)               ,ZETA(100),LABZ(100,10),
     1SIGMA(100),LABS(100,10),W(100),LABW(100,10),ZW(100),LABZW(100,10),
     2AG(350),BG(350),EJ(100),BJ(100),CJ(100),DJ(100)
      DIMENSION BVECTO(100),SVECTO(100),RVECTO(100)
      DIMENSION BVICTO(100),SVICTO(100),RVICTO(100)
      COMMON E,BJ,CJ,DJ
  250 FORMAT(1H1,17HTHE INPUT DATA IS,////)
      DO 1485 N=1,10
      RVICTO(N)=0.0
      SVICTO(N)=0.0
      BVICTO(N)=0.0
      RVECTO(N)=0.0
      SVECTO(N)=0.0
      BVECTO(N)=0.0
 1485 CONTINUE
      WRITE(6,250)
```

```
 200 FORMAT (1X,6A8)
     READ  (5,200) (ITITLE(I),I=1,6)
     READ  (5,200) (ITITLE(I),I=7,12)
     WRITE (6,200) (ITITLE(I),I=1,6)
     WRITE (6,200) (ITITLE(I),I=7,12)
 251 FORMAT(//,8X,2HND,8X,2HNO,8X,2HNZ,8X,2HNS,8X,2HNW,7X,3HNZW,6X
    14HIXUP,3X,7HIYRIGHT,//)
     WRITE(6,251)
 203 FORMAT(8I10)
     READ  (5,203) ND,NO,NZ,NS,NW,NZW,IXUP,IYRITE
     NC=NO+1
     WRITE(6,203) ND,NO,NZ,NS,NW,NZW,IXUP,IYRITE
 463 FORMAT(//,4X,6HIPRINT,//)
     WRITE(6,463)
 464 FORMAT (2I10)
     READ  (5,464) IPRINT,NE
     WRITE (6,464) IPRINT,NE
     IF(NE) 6789,6789,6790
6789 NEE=NE+1
     GO TO 6791
6790 NEE=NE
6791 AIXUP=IXUP
     AIYRIT=IYRITE
 252 FORMAT(//,10X,' LABZ ',//)
     WRITE(6,252)
 205 FORMAT(20A4)
     READ  (5,205)  ((LABZ(M,N),M=1,NZ),N=1,NEE)
     WRITE (6,205)  ((LABZ(M,N),M=1,NZ),N=1,NEE)
 207 FORMAT(//,10X,' LABS ',//)
     WRITE(6,207)
     READ  (5,205)  ((LABS(M,N),M=1,NS),N=1,NEE)
     WRITE (6,205)  ((LABS(M,N),M=1,NS),N=1,NEE)
 208 FORMAT(//,10X,' LABW ',//)
     WRITE(6,208)
     READ  (5,205)  ((LABW(M,N),M=1,NW),N=1,NEE)
     WRITE (6,205)  ((LABW(M,N),M=1,NW),N=1,NEE)
 209 FORMAT(//,10X,' LABZW',//)
     WRITE(6,209)
     READ  (5,205)  ((LABZW(M,N),M=1,NZW),N=1,NEE)
     WRITE (6,205)  ((LABZW(M,N),M=1,NZW),N=1,NEE)
 206 FORMAT(8F10.5)
 210 FORMAT(//,10X,' ZETA ',//)
     WRITE(6,210)
     READ  (5,206)  (ZETA(M), M=1,NZ)
     WRITE (6,206)  (ZETA(M), M=1,NZ)
 872 FORMAT(//,10X,'SIGMA ',//)
     WRITE(6,872)
     READ  (5,206)  (SIGMA(M), M=1,NS)
     WRITE (6,206)  (SIGMA(M), M=1,NS)
```

```
  212 FORMAT(//,10X,' W        ',//)
      WRITE(6,212)
      READ  (5,206)   (W(M),M=1,NW)
      WRITE (6,206)   (W(M),M=1,NW)
  213 FORMAT(//,10X,'ZW        ',//)
      WRITE(6,213)
      READ  (5,206)   (ZW(M),M=1,NZW)
      WRITE (6,206)   (ZW(M),M=1,NZW)
      IF(NE)214,214,9600
  214 WRITE(6,9214)
 9214 FORMAT(//,10X,'CONSTANT COEFFICIENTS IN ASCENDING ORDER',//)
      READ  (5,206)   (DJ(N),N=1,NC)
      WRITE (6,206)   (DJ(N),N=1,NC)
  215 FORMAT(//,10X,'ALPHA     COEFFICIENTS IN ASCENDING ORDER',//)          216
      WRITE(6,215)
      READ  (5,206)   (BJ(N), N=1,NC)
      WRITE (6,206)   (BJ(N), N=1,NC)
  216 FORMAT(//,10X,'BETA      COEFFICIENTS IN ASCENDING ORDER',//)          2
      WRITE(6,216)
      READ  (5,206)   (CJ(N),N=1,NC)
      WRITE (6,206)   (CJ(N),N=1,NC)
      GO TO 23
    6 FORMAT(//,10X,'VALUES OF THE THIRD PARAMETER                ',//)      2
 9600 WRITE(6,6)
      READ (5,206)    (EJ(N),N=1,NE)
      WRITE (6,206)   (EJ(N),N=1,NE)
   23 CONTINUE
  217 FORMAT(//,10X,'INITIAL VALUE OF OMEGA                       ',//)      2
      WRITE(6,217)
  199 FORMAT(E10.5)
      READ  (5,199)   WN
      WRITE (6,199)   WN
  218 FORMAT(//,10X,'      XSCALE                                 ',//)
      WRITE(6,218)
      READ  (5,199)   XSCALE
      WRITE (6,199) XSCALE
  418 FORMAT(//,10X,'      YSCALE                                 ',//)
      WRITE(6,418)
      READ  (5,199)   YSCALE
      WRITE (6,199)   YSCALE
  419 FORMAT(//,10X,'    POWER UP TO WHICH T.DELAY SHOWS     ', //)
      WRITE(6,419)
      READ(5,464)  IPOWER,SCRAP
      WRITE(6,464)  IPOWER,SCRAP
      LL=IPOWER
  420 FORMAT(//,10X,'           TIME-DELAY  CONSTANT          ', //)
      WRITE(6,420)
      READ(5,199)  TDC
      WRITE(6,199)  TDC
```

```
      ROG=.5+AIYRIT
      DAV=.5+AIXUP
      AROG=-ROG*XSCALE
      ADAV=-DAV*YSCALE
      ROGE=9.5-AIYRIT
      DAVE=15.5-AIXUP
      AROGE=ROGE*XSCALE
      ADAVE=DAVE*YSCALE
      FRAN = -AIYRIT *XSCALE
      CHEK =-AIXUP*YSCALE
      IF(NZ) 41,41,708
  708 GO TO(61,62,63,64,65,66,67,68,69,70),ND
   61 G=1.0076
      GO TO 41
   62 G=1.016
      GO TO 41
   63 G=1.0245
      GO TO 41
   64 G=1.0312
      GO TO 41
   65 G=1.0394
      GO TO 41
   66 G=1.0483
      GO TO 41
   67 G=1.0568
      GO TO 41
   68 G=1.0633
      GO TO 41
   69 G=1.071
      GO TO 41
   70 G=1.078
   41 CONTINUE
      AG(1)=0.0
      BG(1)=0.0
      AG(2)=XSCALE
      BG(2)=0.0
      CALL DRAW(2,AG,BG,1,0,LABEL,ITITLE,XSCALE,YSCALE,IXUP,
     1IYRITE ,2,2,9,15,0,LAST)
      IF(NZ) 704,704,705
  705 IF(IPRINT)446,446,9220
  220 FORMAT(   1H1,10X,'CONSTANT ZETA CURVES                    ',//)
 9220 WRITE(6,220)
  527 FORMAT(//,15X,5HALPHA,16X,4HBETA,15X,5HOMEGA,16X,4HZETA,5X,
     115HTHIRD PARAMETER,//)
      WRITE(6,527)
  446 MOD=2
```

```
      DO 4 ME=1,NEE
      E=EJ(ME)
      IF(NE) 8,8,9
    9 CALL COEF
    8 DO 5 M=1,NZ
      J=0
      JG=0
      WNA=WN
      DO 49 L=1,300
      BB1=0.0
      BB2=0.0
      CC1=0.0
      CC2=0.0
      D1=0.0
      D2=0.0
      XXX = TDC*WNA*ZETA(M)
      URTYYY=1-ZETA(M)**2
      RTYYY=SQRT(RTYYY)
      YYY=TDC*WNA*RTYYY
      AMPLIF=EXP(XXX)
      NL=LL+1
      DO 10 N=1,NL
      LLL = N-1
      IF(LLL)2,3,2
    3 XKO=COS(YYY)
      BVECTO(NL+1)=0.0
      XK1=RTYYY*SIN(YYY)-ZETA(M)*COS(YYY)
      SVECTO(1)=-(RTYYY*SIN(YYY)+ZETA(M)*COS(YYY))
    2 XK2=-2.0*ZETA(M)*XK1-XKO
      BB1=BJ(N)*WNA**LLL*XKO+BB1
      CC1=CJ(N)*WNA**LLL*XKO+CC1
      BVECTO(N)=XKO*ZETA(M)
      SVECTO(N+1)=XKO
      RVECTO(N)=BVECTO(N)+SVECTO(N)
      XKO=XK1
      XK1=XK2
   10 CONTINUE
      DO 130 NN=1,NL
      LLL=NN-1
      BB2=BJ(NN)*WNA**LLL*RVECTO(NN)+BB2
      CC2=CJ(NN)*WNA**LLL*RVECTO(NN)+CC2
  130 CONTINUE
      B1=AMPLIF*BB1
      B2=AMPLIF*BB2
      C1=AMPLIF*CC1
      C2=AMPLIF*CC2
      DO 135 NNN=1,NC
      K=NNN-1
      IF(K)136,137,136
```

```
137 TKO=1.0
    BVICTO(NC+1)=0.0
    TK1=-ZETA(M)
    SVICTO(1)=-ZETA(M)
136 TK2=-2.0*ZETA(M)*TK1-TKO
    D1=DJ(NNN)*WNA**K*TKO+D1
    BVICTO(NNN)=TKO*ZETA(M)
    SVICTO(NNN+1)=TKO
    RVICTO(NNN)=BVICTO(NNN)+SVICTO(NNN)
    TKO=TK1
    TK1=TK2
135 CONTINUE
    DO 138  NNNN=1,NC
    K=NNNN-1
    D2=DJ(NNNN)*WNA**K*RVICTO(NNNN)+D2
138 CONTINUE
    Z=1.0E-60
    IF(ABS  (B1*C2-B2*C1)-Z) 11,11,12
 11 GO TO 49
 12 J=J+1
    A(J)=(C1*D2-C2*D1)/(B1*C2-B2*C1)
    B(J)=(B2*D1-B1*D2)/(B1*C2-B2*C1)
    IF(IPRINT)447,447,2000
1001 FORMAT(5E20.5)
2000 WRITE (6,1001)   A(J),B(J),WNA,ZETA(M),E
 447 IF(FRAN-A(J)) 777,777,49
 777 IF(A(J)-AROGE) 778,778,49
 778 IF(CHEK-B(J)) 779,779,49
 779 IF(B(J)-ADAVE) 800,800,49
 800 JG=JG+1
    AG(JG)=A(J)
    BG(JG)=B(J)
 49 WNA=G*WNA
    CALL DRAW(JG,AG,BG,MOD,0,LABZ(M,ME),ITITLE,XSCALE,YSCALE,IXUP,
   1IYRITE ,2,2,9,15,0,LAST)
  5 CONTINUE
  4 CONTINUE
 704 IF(NS) 22,22,601
 601 IF(IPRINT)448,448,9221
9221 WRITE(6,221)
 221 FORMAT(    1H1,10X,'CONSTANT SIGMA CURVES                 ',//)
 222 FORMAT(15X,5HALPHA,16X,4HBETA,15X,5HSIGMA,5X,15HTHIRD PARAMETER,/
   1/)
    WRITE(6,222)
 448 DO 7 ME=1,NEE
    E=EJ(ME)
    IF (NE) 13,13,14
 14 CALL COEF
 13 DO 22  M=1,NS
```

```
      DD=0.0
      CC1=0.0
      BB1=0.0
      EXPAMP=-TDC*SIGMA(M)
      AMPLIF=EXP(EXPAMP)
      DO 21 N=1,NC
      K=N-1
      DUMMY5=0.0
      IF(SIGMA(M).NE.0.0) DUMMY5=SIGMA(M)**K
      DD=(-1.0)**K*DJ(N)*DUMMY5+DD
   21 CONTINUE
      NL=LL+1
      DO 181 NN=1,NL
      LLL=NN-1
      DUMMY5=0.0
      IF(SIGMA(M).NE.0.0) DUMMY5=SIGMA(M)**K
      CC1=(-1.0)**LLL*CJ(NN)*DUMMY5+CC1
      BB1=(-1.0)**LLL*BJ(NN)*DUMMY5+BB1
  181 CONTINUE
      BB=AMPLIF*BB1
      CC=AMPLIF*CC1
      J=1
      A(J)=-DD/BB
      B(J)=0.0
      IF(IPRINT)449,449,9002
 9002 WRITE (6,1002)  A(J), B(J), SIGMA(M),E
 1002 FORMAT(4E20.5)
  449 IF(AROG-A(J)) 110,110,310
  110 IF(A(J)-AROGE) 111,111,310
  111 J=J+1
  310 A(J)=0.0
      B(J)=-DD/CC
      IF(IPRINT)450,450,451
  451 WRITE (6,1002)  A(J), B(J), SIGMA(M),E
  450 IF(ADAV-B(J)) 112,112,311
  112 IF(B(J)-ADAVE) 113,113,311
  113 J=J+1
  311 B(J)=(15.0-AIXUP)*YSCALE
      A(J)=(-CC*B(J)-DD)/BB
      IF(IPRINT)452,452,453
  453 WRITE (6,1002)  A(J), B(J), SIGMA(M),E
  452 IF(AROG-A(J)) 114,114,312
  114 IF(A(J)-AROGE) 117,117,312
  312 A(J)=(9.0-AIYRIT )*XSCALE
      B(J)=(-BB*A(J)-DD)/CC
      IF(IPRINT)454,454,455
```

```
455 WRITE (6,4002)  A(J), B(J), SIGMA(M),E
454 IF(ADAV-B(J)) 116,116,118
116 IF(B(J)-ADAVE) 117,117,118
118 J=J-1
117 CALL DRAW(J,A,B,2,0,LABS(M,ME),ITITLE,XSCALE,YSCALE,IXUP,IYRITE
   1,2,2,9,15,0,LAST)
 22 CONTINUE
  7 CONTINUE
    IF(N7W)9999,9999,602
602 IF(IPRINT)456,456,9225
9225 WRITE(6,225)
225 FORMAT(1H1,26HCONSTANT ZETA-OMEGA CURVES,//)
226 FORMAT(15X,5HALPHA,16X,4HBETA,10X,10HZETA-OMEGA,5X,
   115HTHIRD PARAMETER,//)
    WRITE(6,226)
456 DO 15 ME=1,NEE
    E=EJ(ME)
    IF(NE) 16,16,17
 17 CALL COEF
 16 DO 31 M=1,NZW
    J=0
    JG=0
    AZETA=.00333
    DO 35 L=1,299
    WN=ZW(M)/AZETA
    D1=0.0
    D2=0.0
    C1=0.0
    C2=0.0
    B1=0.0
    B2=0.0
    DO 32 N=1,NC
    K=N-1
    IF(K) 33,34,33
 34 Q1=0.0
    Q=-1.0/WN**2
 33 D2=DJ(N)*Q1+D2
     C2=CJ(N)*Q1+C2
    B2=BJ(N)*Q1+B2
    D1=DJ(N)*Q+D1
    C1=CJ(N)*Q+C1
    B1=BJ(N)*Q+B1
    Q2=-2.0*ZW(M)*Q1-WN**2*Q
    Q=Q1
 32 Q1=Q2
    IF(ABS (B1*C2-B2*C1)-Z) 35,35,29
 29 J=J+1
    A(J)=(C1*D2-C2*D1)/(B1*C2-B2*C1)
    B(J)=(B2*D1-B1*D2)/(B1*C2-B2*C1)
```

10

```
      IF(IPRINT)457,457,458
  458 WRITE (6,1002)  A(J), B(J), ZW(M)    ,E
  457 IF(FRAN-A(J)) 104,104,35
  104 IF(A(J)-AROGE) 105,105,35
  105 IF(CHEK-B(J)) 106,106,35
  106 IF(B(J)-ADAVE) 107,107,35
  107 JG=JG+1
      AG(JG)=A(J)
      BG(JG)=B(J)
   35 AZETA=AZETA+.00333
   37 CALL DRAW(JG,AG,BG,2,0,LABZW(M,ME),ITITLE,XSCALE,YSCALE,IXUP,
     1IYRITE ,2,2.9,15,0,LAST)
   31 CONTINUE
   15 CONTINUE
 9999 IF(NW)1006,1006,702
  702 IF(IPRINT)459,459,9223
 9223 WRITE(6,223)
  223 FORMAT(1H1,21HCONSTANT OMEGA CURVES,//)
  224 FORMAT(15X,5HALPHA,16X,4HBETA,15X,5HOMEGA,15X,5HAZETA, 5X,
     115HTHIRD PARAMETER,//)
      WRITE(6,224)
  459 DO 18 ME=1,NEE
      E=EJ(ME)
      IF(NF) 19,19,20
   20 CALL COEF
   19 DO 24 M=1,NW
      J=0
      JG=0
      AZETA=0.0
      DO 25 L=1,300
      BB1=0.0
      BB2=0.0
      CC1=0.0
      CC2=0.0
      D1=0.0
      D2=0.0
      XXX = TDC * AZETA*W(M)
      URTYYY=1-AZETA**2
      RTYYY= SQRT(RTYYY)
      YYY = TDC*W(M)*RTYYY
      AMPLIF=EXP(XXX)
      NL = LL+1
      DO 26 N=1,NL
      LLL=N-1
      IF(LLL)27,28,27
   28 XKO=COS(YYY)
      BVECTO(NL+1)=0.0
      XK1=RTYYY*SIN(YYY)-AZETA*COS(YYY)
```

```
      SVECTO(1)=-(RTYYY*SIN(YYY)+AZETA*COS(YYY))
   27 XK2=-2.0*AZETA*XK1-XKO
      BB1=BJ(N)*W(M)**LLL*XKO+BB1
      CC1=CJ(N)*W(M)**LLL*XKO+CC1
      BVECTO(N)=XKO*AZETA
      SVECTO(N+1)=XKO
      RVECTO(N)=BVECTO(N)+SVECTO(N)
      XKO=XK1
      XK1=XK2
   26 CONTINUE
      DO150 NN=1.NL
      LLL = NN-1
      BB2=BJ(NN)*W(M)**LLL*RVECTO(NN)+BB2
      CC2=CJ(NN)*W(M)**LLL*RVECTO(NN)+CC2
  150 CONTINUE
      B1=AMPLIF*BB1
      B2=AMPLIF*BB2
      C1=AMPLIF*CC1
      C2=AMPLIF*CC2
      DO 155 NNN=1,NC
      K=NNN-1
      IF(K)156,157,156
  157 TKO=1.0
      BVICTO(NC+1)=0.0
      TK1=-AZETA
      SVICTO(1)=-AZETA
  156 TK2=-2.0*AZETA*TK1-TKO
      D1=DJ(NNN)*W(M)**K*TKO+D1
      BVICTO(NNN)=TKO*AZETA
      SVICTO(NNN+1)=TKO
      RVICTO(NNN)=BVICTO(NNN)+SVICTO(NNN)
      TKO=TK1
      TK1=TK2
  155 CONTINUE
      DO 158 NNNN=1,NC
      K=NNNN-1
      D2=DJ(NNNN)*W(M)**K*RVICTO(NNNN)+D2
  158 CONTINUE
      IF(ABS (B1*C2-B2*C1)-Z)25,25,30
   30 J=J+1
      A(J)=(C1*D2-C2*D1)/(B1*C2-B2*C1)
      B(J)=(B2*D1-B1*D2)/(B1*C2-B2*C1)
      IF(IPRINT)460,460,461
  461 WRITE (6,1001) A(J), B(J),W(M), AZETA ,E
  460 IF(FRAN-A(J)) 100,100,25
  100 IF(A(J)-AROGE) 101,101,25
  101 IF(CHEK-B(J)) 102,102,25
```

12

```
 102 IF(B(J)-ADAVE) 103,103,25
 103 JG=JG+1
     AG(JG)=A(J)
     BG(JG)=B(J)
  25 AZETA=AZETA+.00333
  24 CALL DRAW(JG,AG,BG,MOD,0,LABW(M,ME),ITITLE,XSCALE,YSCALE,IXUP,
    1IYRITE,2,2,9,15,0,LAST)
  18 CONTINUE
1006 AG(1)=0.0
     BG(1)=0.0
     AG(2)=XSCALE
     BG(2)=0.0
     CALL DRAW(2,AG,BG,3,0,LABEL,ITITLE,XSCALE,YSCALE,IXUP,
    1IYRITE,2,2,9,15,0,LAST)
     GO TO 1485
     END
```

```
SUBROUTINE COEF

DIMENSION BJ(100),CJ(100),DJ(100)
COMMON E,BJ,CJ,DJ
CJ(1)=0.488
DJ(1)=-147.8
BJ(1)=0.0
CJ(2)=0.444
DJ(2)=3165.6
BJ(2)=0.488
CJ(3)=1.0
DJ(3)=5087.4
BJ(3)=0.444
CJ(4)=0.0
DJ(4)=8973.66
BJ(4)=1.0
CJ(5)=0.0
DJ(5)=4941.3
BJ(5)=0.0
CJ(6)=0.0
DJ(6)=98.46
BJ(6)=0.0
CJ(7)=0.0
DJ(7)=1.0
BJ(7)=0.0
RETURN
END
```

APPENDIX E

SOLUTION TO A QUARTIC EQUATION

1. A quartic equation

$$X^4 + aX^3 + bX^2 + cX + d \tag{E-1}$$

has the resolvent cubic equation

$$Y^3 - bY^2 + (ac-4d)Y - a^2d + 4bd - c^2 = 0 \tag{E-2}$$

2. By using the attached solution to a cubic equation, one can specify arbitrarily a solution $y = y_r$, to equation (E-2) and then define

$$R = \sqrt{\frac{a^2}{4} - b + y_r}$$

3. The four roots to (E-1), will then be:

$$x = -\frac{a}{4} + \frac{R}{2} \pm \frac{D}{2}$$

$$x = -\frac{a}{4} - \frac{R}{2} \pm \frac{E}{2}$$

4. The parameters E and D, take the following values:

a. $R \neq 0$

$$D = \sqrt{\frac{3a^2}{4} - R^2 - 2b + \frac{4ab-8c-a^3}{4R}}$$

$$E = \sqrt{\frac{3a^2}{4} - R^2 - 2b - \frac{4ab-8c-a^3}{4R}}$$

b. $R = 0$

$$D = \sqrt{\frac{3a^2}{4} - 2b + 2\sqrt{y_r^2 - 4d}}$$

$$E = \sqrt{\frac{3a^2}{4} - 2b - 2\sqrt{y_r^2 - 4d}}$$

ATTACHMENT TO APPENDIX E

SOLUTION TO A CUBIC EQUATION

1. Consider the Cubic:

$$Y^3 + pY^2 + qY + r = 0 \qquad \text{(E-1)}$$

Let: $y = x - \frac{p}{3}$. (E-2)

Calculate:

$$y^2 = x^2 - \frac{2px}{3} + \frac{p^2}{9}$$

$$y^3 = x^3 - x^2 p + \frac{xp^2}{3} - \frac{p^3}{27}$$

and substitute into (E-1) , it follows:

$$x^3 + x(q - \frac{p^2}{3}) + \frac{2p^3}{27} - \frac{pq}{3} + r$$

2. The cubic is now in the form:

$x^3 + ax + b = 0$, where $a = q - \frac{p^2}{3}$, and $b = \frac{2p^3}{27} - \frac{pq}{3} + r$.

3. Letting:

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \quad , \quad M = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

The three solutions for y come out to be:

$$y_1 = A + M - \frac{p}{3}$$

$$y_2 = -\frac{A+M}{2} + \frac{A-M}{2}\sqrt{-3}$$

$$y_3 = -\frac{A+M}{2} - \frac{A-M}{2}\sqrt{-3}$$

LIST OF REFERENCES

Reference Number | Source

1. Siljak, D.D., Nonlinear Systems, pp. 1-55, Wiley,1969.

2. Hollister, F.H., Network Analysis and Design by Parameter Plane Techniques, pp. 14-18, Thesis research, USNPGS 1965.

3. Holst, P.A.,"Padé Approximations and Analog Computer Simulations of Time Delays," Simulation, June 1969, pp. 277-280.

4. Eisenberg, L., Design of Feed-Back Control Systems With Transport Lag by Parameter Plane Techniques, pp. 59-66, Doctoral Dissertation at Newark College of Engineering, 1966.

5. Lamberski, S.E., Parameter Plane Solution of Linear Feed-Back Control Systems with Alpha-Beta Product and Transport Lag, pp. 2-11, Thesis research at Newark College of Engineering, 1967.

6. Karmarkar, J.S., "Stability Analysis of Systems With Time Delay" pp. 1421-1424, Proc. IEE, v. 117, No. 7, July 1970.

7. Siljak, D.D., Nonlinear Systems, pp. 392 - 396, p. 455, Wiley, 1969.

8. Pontryagin, L.S., "On the Zeros of Some Elementary Transcendental Functions," pp. 103-110, American Math. Soc.,Trans.,1955.

9. Ince, E.L., Principles of Descriptive Geometry, Longmans, Green, 1933.

10. Cadena, J. E., and Thaler, G. J.,"Characteristics of Three Parameter Space with Applications to System Synthesis," Proceedings of the Eighth International Symposium On Space Technology and Science.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Professor G. J. Thaler, Code 52Tr Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	1
4.	Professor A. Gerba, Code 52Gz Department of Electrical Engineering Naval Postgraduate School Monterey, California 93940	1
5.	Hellenic Naval Command Third E.G./7 Athens, Greece	1
6.	Chairman, Electrical Engineering Dept. National Technical University Athens, Greece	1
7.	Chairman, Chemical Engineering Dept. National Technical University Athens, Greece	1
8.	Admiral K. Pezopoulos, Professor (ret.) National Technical University c/o LCDR A. Hadjiconstandis, H.N. Koritsas 39, Papagou Athens, Greece	1
9.	Admiral CH. Hadjiconstandis (ret.) c/o LCDR A. Hadjiconstandis, H.N. Koritsas 39, Papagou Athens, Greece	1
10.	LCDR A. Hadjiconstandis, H. N. Koritsas 39, Papagou Athens, Greece	1

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

...GATING ACTIVITY (Corporate author) Na...al Postgraduate School Mo...terey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

...RT TITLE

Pa...ameter Plane Methods for Linear Systems With Time Delay

...SCRIPTIVE NOTES (Type of report and inclusive dates)

Ma...ter's Thesis; March 1973

...THOR(S) (First name, middle initial, last name)

An...hony .Ch. Hadjiconstandis

...RT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
Ma...ch 1973	183	10

...TRACT OR GRANT NO. ...OJECT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

...STRIBUTION STATEMENT

Approved for public release; distribution unlimited.

...PPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940

...STRACT

After a brief introduction on the parameter plane methods, the behavior of simple systems with time delay has been initially studied by the root-loci method and then the state of the art of parameter plane methods with time delay has been presented.

A variation of the solution in the state of the art presentation has been derived and an appropriate digital computer program has been constructed. As an extension of studies in the parameter plane, a three variable parameter system has been approached by using basic descriptive geometry properties and simple illustrative examples have been worked out in the parameter space.

Finally, a real life system with time delay has been analysed and redesigned by parameter plane methods.

DD FORM 1 NOV 65 1473 (PAGE 1)

01-807-6811

UNCLASSIFIED

Security Classification

A-31408

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Parameter Plane Analysis						
Time Delay						
Control Theory						

Thesis 143545
H1095 Hadjiconstandis
c.1 Parameter plane methods for linear systems with time delay.